The Ising Model

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1 Finite-volume Gibbs distributions

In this section, the Ising model on $\mathbb{Z}^d$ is defined precisely and some of its basic properties are established.

Definition 1.0 Gibbs distribution in finite-volume

For Ising model in a finite volume $\Lambda \subset \mathbb{Z}^d$, the configurations of the Ising model are the elements of the set $\Omega_\Lambda = \{-1, 1\}^\Lambda$, with the random variable **spin** $\sigma_i(\omega) = \omega_i$.
we can define the Gibbs distribution in steps:
For the given edge set $\mathscr{E}_\Lambda$, we associate each configuration $\omega \in \Omega_\Lambda$ with its **energy**, given by the Hamiltonian

$$\mathscr{H}_{\Lambda;\beta,h}(\omega) = -\beta \sum_{\{i,j\}\in\mathscr{E}_\Lambda} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i\in\Lambda} \sigma_i(\omega),$$

where $\beta \in \mathbb{R}_{\geq 0}$ is the inverse temperature and $h \in \mathbb{R}$ is the magnetic field. Then the Gibbs distribution is defined by

$$\mu_{\Lambda;\beta,h}(\omega) = \frac{1}{Z_{\Lambda;\beta,h}} \exp(-\mathscr{H}_{\Lambda;\beta,h}(\omega)),$$

where the partition function is given by

$$Z_{\Lambda;\beta,h} = \sum_{\omega\in\Omega_\Lambda} \exp(-\mathscr{H}_{\Lambda;\beta,h}(\omega)).$$

In this Chapter, the edge set is usually given by the nearest-neighbor pairs of vertices.

Example 1.0

With the definition above we introduce several important concepts in the Ising model:

- Free boundary condition: $\mathscr{E}_\Lambda = \{\{i,j\} \in \mathbb{Z}^d \times \mathbb{Z}^d : |i-j| = 1, i, j \in \Lambda\}$. Denoted as $\mu_{\Lambda;\beta,\varnothing}$
- Periodic boundary condition: $\Lambda_n = \{0, \ldots, n-1\}^d, \mathscr{E}_{\Lambda_n} = \{\{i,j\} : i - j$ has only one nonzero component and the latter is equal to $\pm 1 \bmod n\}$ (Ising model on the torus $\mathbb{T}_n$). Denoted as $\mu^{\text{per}}_{\Lambda_n;\beta,h}$
- Boundary condition: Fixing a finite set $\Lambda \subset \mathbb{Z}^d$ and a configuration $\eta \in \Omega$, we define a configuration of the Ising model in Λ with boundary condition η as an element of the

finite set

$$\Omega_\Lambda^\eta = \{\omega \in \Omega : \omega_i = \eta_i, \forall i \notin \Lambda\}.$$

with the edge set

$$\mathscr{E}_\Lambda^b = \{\{i,j\} \subset \mathbb{Z}^d : \{i,j\} \cap \Lambda \neq, i \sim j\}.$$

Denoted as $\mu^\eta_{\Lambda;\beta,h}$

- For special case of $\eta \equiv 1$ (all spins up), we denote the Gibbs distribution as $\mu^+_{\Lambda;\beta,h}$. For $\eta \equiv -1$ (all spins down), we denote it as $\mu^-_{\Lambda;\beta,h}$.

Remark 1.1. We will often use $\langle \cdot \rangle^{\#}_{\Lambda;\beta,h}$ and $\mu^{\#}_{\Lambda;\beta,h}$ interchangeably.

2 Thermodynamic limit, pressure and magnetization

2.1 Convergence of subsets

To define the Ising model on the whole lattice $\mathbb{Z}^d$, we have to consider sequences of finite subsets $\Lambda_n \subset \mathbb{Z}^d$ which **converge to** $\mathbb{Z}^d$, denoted by $\Lambda_n \uparrow \mathbb{Z}^d$:

- $\Lambda_n \subset \Lambda_{n+1}$ for all $n \in \mathbb{N}$.
- $\bigcup_{n=1}^\infty \Lambda_n = \mathbb{Z}^d$.

Furthermore, in order to control the influence of the boundary condition, we have to impose a further regularity. We'll say that a sequence Λ_n **converges to** $\mathbb{Z}^d$ **in the sense of van Hove**, which is denoted as $\Lambda_n \Uparrow \mathbb{Z}^d$ iff

$$\lim_{n\to\infty} \frac{|\partial^{in}\Lambda_n|}{|\Lambda_n|} = 0, \tag{1}$$

where $\partial^{in}\Lambda = \{i \in \Lambda : \exists j \notin \Lambda, j \sim i\}$.

2.2 Pressure

Definition 2.0 Pressure

The **pressre** in Λ is defined by

$$\psi^{\#}_\Lambda(\beta,h) = \frac{1}{|\Lambda|}\log Z^{\#}_{\Lambda;\beta,h}.$$

Remark 2.1. We can observe that it is an even function of h.

Lemma 2.1

For each type of boundary condition $\#$, $(\beta,h) \to \psi^{\#}_\Lambda(\beta,h)$ is convex.

Proof

By Holder's inequality,

$$Z_{\Lambda;\alpha\beta_1+(1-\alpha)\beta_2,\alpha h_1+(1-\alpha)h_2} = \sum_{\omega} e^{-\alpha\mathscr{H}_{\Lambda;\beta_1,h_1}(\omega)-(1-\alpha)\mathscr{H}_{\Lambda;\beta_2,h_2}(\omega)}$$
$$\leq (\sum_{\omega} e^{-\mathscr{H}_{\Lambda;\beta_1,h_1(\omega)}})^{\alpha}(\sum_{\omega} e^{-\mathscr{H}_{\Lambda;\beta_2,h_2}(\omega)})^{1-\alpha}.$$

Thus it is convex by taking logarithm and dividing by $|\Lambda|$.

Theorem 2.1

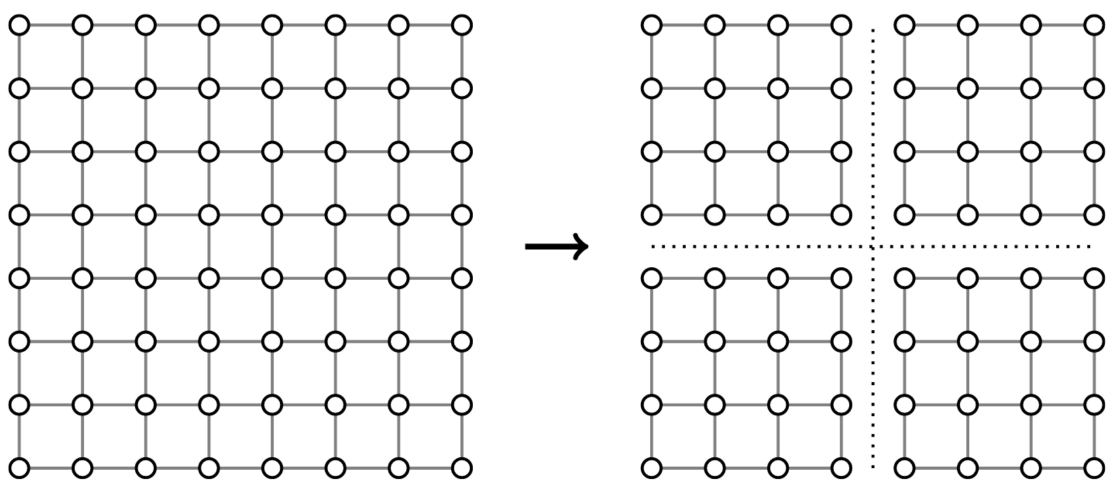
In the thermodynamic limit, the **pressure**

$$\psi(\beta,h) = \lim_{\Lambda\Uparrow\mathbb{Z}^d} \psi_{\Lambda}^{\#}(\beta,h)$$

is well-defined and independent of the sequence and of the type of boundary condition.

Proof

We start by proving convergence in the case of free boundary condition.
Step 1: We first show the existence of the limit $\lim_{n\to\infty} \psi^{\varnothing}_{D_n}(\beta,h)$, where $D_n = \{1,2,\dots 2^n\}^d$. The pressure of D_{n+1} will be shown to be close to the pressure of D_n. Indeed, we can decompose D_{n+1} into 2^d disjoint translates of D_n, denoted by $D_n^{(1)},\dots,D_n^{(2^d)}$.



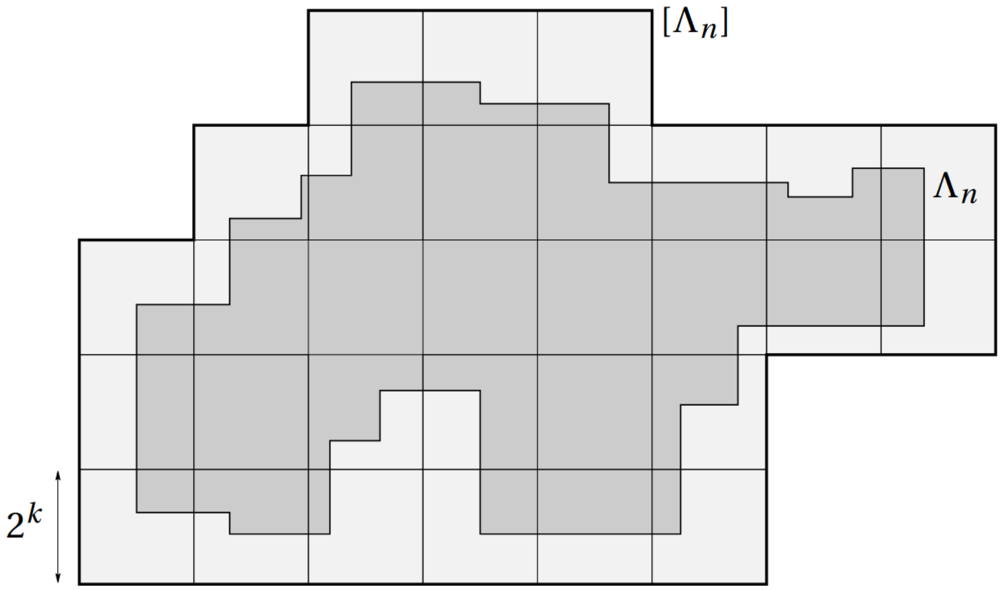
Thus the energy of ω in $\mathbb{D}_{n+1}$ can be written as

$$\mathscr{H}_{D_{n+1}} = \sum_{i=1}^{2^d} \mathscr{H}_{D_n^{(i)}} + R_n,$$

where R_n represents the energy of interaction between pairs of spins that belong to different sub-boxes. Since $\sum_\omega \prod_{i=1}^{2^d} \exp\Big(-\mathscr{H}_{D_n^{(i)}(\omega)}\Big) = (Z_{D_n})^{2^d}$, we have that

$$e^{-\sup R_n}(Z_{D_n})^{2^d} \le Z_{D_{n+1}} \le e^{-\inf R_n}(Z_{D_n})^{2^d},$$

i.e. $|\psi_{D_{n+1}} - \psi_{D_n}| \le \frac{\sup |R_n|}{2^{d(n+1)}}$. Since $|R_n| \le \beta \cdot d \cdot (2^{(n+1)(d-1)})$, we have $|\psi_{D_{n+1}} - \psi_{D_n}| \le Ce^{-cn}$, which implies that it is a Cauchy sequence.
Step 2: We now consider an arbitrary sequence $\Lambda_n \Uparrow \mathbb{Z}^d$. We fix some integer k and consider a partition of $\mathbb{Z}^d$ into adjacent disjoint translates of D_k. For each n, consider a minimal covering of Λ_n by elements $D_k^{(j)}$ of the partition, and let $[\Lambda_n] = \bigcup_j D_k^{(j)}$.



We use the estimate

$$|\psi_{\Lambda_n} - \psi| \le |\psi_{\Lambda_n} - \psi_{[\Lambda_n]}| + |\psi_{[\Lambda_n]} - \psi_{D_k}| + |\psi_{D_k}| \tag{2}$$

With step 1, we have that $\exists k_0$ big enough such that for all $k \ge k_0$, $|\psi_{D_k} - \psi| < \epsilon$.
For $\psi_{[\Lambda_n]}$, we have $\mathscr{H}_{[\Lambda_n]} = \sum_j \mathscr{H}_{D_k^{(j)}} + W_n$, where $|W_n| \le \beta \frac{|[\Lambda_n]|}{D_k} d(2^k)^{d-1} = \beta d 2^{-k} |[\Lambda_n]|$ (counting the number of sub-boxes). Thus $\exists k_1$ big enough such that for all $k \ge k_1$, $|\psi_{[\Lambda_n]} - \psi_{D_k}| < \epsilon$.
Then we fix $k \ge \max\{k_0, k_1\}$ and write $\Delta_n = [\Lambda_n] \setminus \Lambda_n$. We have

$$|\mathscr{H}_{\Lambda_n} - \mathscr{H}_{[\Lambda_n]}| \le (2d\beta + |h|)|\Delta_n|.$$

hence

$$e^{-(2d\beta + |h| + \log_2)|\Delta_n|} Z_{\Lambda_n} \le Z_{[\Lambda_n]} \le e^{(2d\beta + |h| + \log_2)|\Delta_n|} Z_{\Lambda_n}$$

i.e.

$$|\log Z_{\Lambda_n} - \log Z_{[\Lambda_n]}| \le |\partial \Lambda_n||D_k|(2d\beta + |h| + \log_2)$$

Since $\frac{|[\Lambda_n]|}{|\Lambda_n|} \in [1, 1 + \frac{|\partial^{in}\Lambda_n||D_k|}{|\Lambda_n|}]$ and ψ_Λ is uniformly bounded by $2d\beta + |h| + \log 2$, we can find n large enough such that $|\psi_{\Lambda_n} - \psi| < \varepsilon$ from the regularity (1).

Combining all these estimates, we conclude from (2) that $|\psi^{\varnothing}_{\Lambda_n} - \psi| \le \varepsilon$.
The independence of boundary condition can get in the similar estimation from the regularity.

2.3 Magnetization

Definition 2.1 magnetization density

We define the **magnetization density** as

$$m_\Lambda = \frac{1}{|\Lambda|} M_\Lambda,$$

where $M_\Lambda = \sum_{i\in\Lambda} \sigma_i$ is the total magnetization in Λ.

Remark 2.2. As can be easily checked,

$$m^{\#}_\Lambda(\beta,h) = \frac{\partial \psi^{\#}_\Lambda}{\partial h}(\beta,h).$$

Learned in Probability Theory-Outer Chapter, we have also learned that

$$\log \left\langle e^{tM_\Lambda} \right\rangle^{\#}_{\Lambda;\beta,h} = |\Lambda|(\psi^{\#}_\Lambda(\beta,h+t) - \psi^{\#}_\Lambda(\beta,h)). \tag{3}$$

It will turn out to be important to determine whether the equation above still holds in the thermodynamic limit. We need to check whether the limit exists and depends on the boundary condition, and if we can interchange the limit and the derivative.

Since $\psi^{\#}_\Lambda(\beta,h)$ is convex, we can use the following lemma.

Lemma 2.2 Properties of convex functions

(1) $\partial^+ f(x), \partial^- f(x)$ exist at all points $x \in I$.

(2) $\partial^- f(x) \le \partial^+ f(x)$ for all $x \in I$.

(3) $\partial^+ f(x) \le \partial^- f(y)$ for all $x < y \in I$.

(4) $\partial^+ f, \partial^- f$ are nondecreasing.

(5) $\partial^+ f$ is right-continuous, $\partial^- f$ is left-continuous.

(6) $\{x : \partial^+ f(x) \ne \partial^- f(x)\}$ is at most countable.

(7) Let (g_n) be a sequence of convex functions from I to $\mathbb{R}$ converging pointwise to a function g. If g is differentiable at x, then $\lim_{n\to\infty} \partial^+ g_n(x) = \lim_{n\to\infty} \partial^- g_n(x) = g'(x)$.

With the lemma above, we immediately reach that

Theorem 2.2

Define $\mathfrak{B}_\beta = \{h \in \mathbb{R} : \frac{\partial\psi}{\partial h^-}(\beta,h) \ne \frac{\partial\psi}{\partial h^+}(\beta,h)\}$.

Then for all $h \notin \mathfrak{B}_\beta$, the **average magnetization density**

$$m(\beta, h) = \lim_{\Lambda \Uparrow \mathbb{Z}^d} m_\Lambda^\#(\beta, h)$$

is well defined, independent of the sequence $\Lambda \Uparrow \mathbb{Z}^d$ and of the boundary condition and satisfies

$$m(\beta, h) = \frac{\partial \psi}{\partial h}(\beta, h). \tag{4}$$

Moreover, the function $h \to m(\beta, h)$ is non-decreasing and is continuous at every $h \notin \mathfrak{b}_\beta$. It is however discontinuous at each $h \in \mathfrak{b}_\beta$.
In particular, the **spontaneous magnetization**

$$m^*(\beta) = \lim_{h \downarrow 0} m(\beta, h)$$

is always well-defined.

The above discussion shows that the average magnetization density is discontinuous precisely when the pressure is not differentiable in h. This leads to the following

Definition 2.2

The pressure psi exhibits a **first-order phase transition** at (β, h) if $h \mapsto \psi(\beta, h)$ fails to be differentiable at that point.

3 The one-dimensional Ising model

Since the major part of this section has been discussed in Probability Theory-Outer Chapter, I'll only introduce them briefly.

Theorem 3.0

$(d = 1)$ For all $\beta \geq 0$ and all $h \in \mathbb{R}$, the pressure $\psi(\beta, h)$ of the one-dimensional Ising model is given by

$$\psi(\beta, h) = \log \left\{ e^\beta \cosh(h) + \sqrt{e^{2\beta} \cosh^2(h) - 2\sinh(2\beta)} \right\}. \tag{5}$$

Remark 3.1. With the theorem above, we can check that

- $m^*(beta) = 0, \quad \forall \beta > 0.$
- $\lim_{h \to \pm\infty} m(\beta, h) = \pm 1, \forall \beta \geq 0.$
- $\lim_{\beta \to \infty} m(\beta, h) = \begin{cases} +1, & h > 0 \\ 0, & h = 0 \\ -1, & h < 0 \end{cases}.$

Theorem 3.1 Exponential decay

($d = 1$) Let $0 < \beta < \infty$ and consider any sequence $\Lambda_n \Uparrow \mathbb{Z}$, with an arbitrary boundary condition $\#$. For all $\varepsilon > 0$, there exists $c = c(\beta, \varepsilon) > 0$ such that, for large enough n,

$$\mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \notin (-\varepsilon, \varepsilon)) \leq e^{-c|\Lambda_n|}. \tag{6}$$

Proof

We start by writing

$$\mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \notin (-\varepsilon, \varepsilon)) = \mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \geq \varepsilon) + \mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \leq -\varepsilon),$$

which can be studied in the same way.

$$\mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \geq \varepsilon) \leq e^{-h\varepsilon|\Lambda_n|} \left\langle e^{hm_{\Lambda_n}|\Lambda_n|} \right\rangle.$$

Since $\left\langle e^{hm_{\Lambda_n}|\Lambda_n|} \right\rangle = Z_{\Lambda_n;\beta,h}/Z_{\Lambda_n;\beta,0}$, we have

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \log \mu_{\Lambda_n;\beta,0}(m_{\Lambda_n} \geq \varepsilon) \leq \lim_{n\to\infty} (\psi_{\Lambda_n}(\beta, h) - \psi_{\Lambda_n}(\beta, 0) - h\varepsilon) = I_\beta(h) - h\varepsilon,$$

where $I_\beta(h) = \psi(\beta, h) - \psi(\beta, 0)$. Since $h \geq 0$ was arbitrary, we can minimize over tha latter:

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \log \mu_{\Lambda_n;\beta,0}(m_{\Lambda_n} \geq \varepsilon) \leq -\sup\{h\varepsilon - I_\beta(h)\}.$$

We suffice to show that $\sup\{h\varepsilon - I - \beta(h)\} > 0$. Since $I'_\beta(0) = 0, I'_\beta(h) \to 1$ as $h \to \infty$, therefore, for each $0 < \varepsilon < 1$, there exists some $h_* > 0$ such that $h_*\varepsilon - I_\beta(h_*) > 0$.

4 Infinite-volume Gibbs states

The pressure only provides information about the thermodynamical behavior of the system in large volumes. If one is interested in the statistical properties of general observales, such as the fluctuations of the magnetization density in a finite region or the correlations between far apart spins, one needs to understand the behaviour of the Gibbs distribution $\mu_{\Lambda;\beta,h}$ in large volumes.

In this chapter, we will follow a hands-on approach: a state (in infinite volume) will be identified with an assignment of an average value to each local function.

Definition 4.0

A function $f : \Omega \to \mathbb{R}$ is **local** if there exists $\Delta \subset \mathbb{Z}^d$ such that $f(\omega) = f(\omega')$ as sonn as ω and ω' coincide on Δ. The smallest such set Δ is called the **support** of f and is denoted by $\text{supp}(f)$.

Definition 4.0

An **infinite-vollume state** is a mapping associating to each local function f a real number $\langle f \rangle$ and satisfying:

- Normalization: $\langle 1 \rangle = 1$.
- Positivity: $f \geq 0 \Rightarrow \langle f \rangle \geq 0$.
- Linearity: $\forall \lambda \in \mathbb{R}, \langle f + \lambda g \rangle = \langle f \rangle + \lambda \langle g \rangle$.

The number $\langle f \rangle$ is called the average of f in the state $\langle \cdot \rangle$.

Since states are defined on the infinite lattice, it is natural to distinguish thos that are translation invariant. The **translation** by $j \in \mathbb{Z}^d, \theta_j : \mathbb{Z}^d \to \mathbb{Z}^d$ is defined by

$$\theta_j i = i + j.$$

Definition 4.0

A state $\langle \cdot \rangle$ is **translation invariant** if $\langle \theta_j f \rangle = \langle f \rangle$ for all $j \in \mathbb{Z}^d$ and all local functions f.

Theorem 4.0

Let $\beta \geq 0$ and $h \in \mathbb{R}$. Along any sequence $\Lambda_n \uparrow \mathbb{Z}^d$, the finite-volume Gibbs distributions with $+$ or $-$ boundary condition converge to infinite-volume Gibbs states $\langle \cdot \rangle^+_{\beta,h}, \langle \cdot \rangle^-_{\beta,h}$. The states do not depend on the sequence (Λ_n) and are both translation invariant.

The prrof will be given in 3.6 (Not included in this note).

Remark 4.1. Notice that we do not claim that the two states are distinct.

More generally, one can prove, albeit in a non-constructive way, that any sequence of finite-volume Gibbs distributions admits converging subsequences.

Exercise 4.1

Let (η_n) be a sequence of boundary conditions and $\Lambda_n \uparrow \mathbb{Z}^d$. Prove that there exists an increasing sequence n_k of integers and a Gibbs state $\langle \cdot \rangle$ such that

$$\langle \cdot \rangle = \lim_{k \to \infty} \langle \cdot \rangle^{\eta_{n_k}}_{\Lambda_{n_k};\beta,h}$$

is well defined.

Proof

Take $\{n^{(0)}_k = k\}$.
We do by induction and assume $m = 1$ at first. Consider all local events whose support is contained in Λ_m. Since the events rely on finite-many configurations, we need only to consider the expectation of r.v.s I_A, where $A \subset \Lambda_m$.

Now since the sequence $\{\langle I_A\rangle_{\Lambda_{n_k^{(m-1)}};\beta,h}\}$ is bounded by 1, we are able to find a subsequence $\{n_k^{(m)}\}$ of $\{n_k^{(m-1)}\}$ such that $\lim_{k\to\infty}\langle I_A\rangle_{\Lambda_{n_k^{(m)}}}$ exists for all $A\subset\Lambda_m$.

We now take the subsequence $\{n_k^{(k)}\}$, it's exactly the subsequence we want.

5 Two families of local functions

As seen in the previous exercise, we need only to test convergence on a restricted family of functions. The following lemma provides two particularly convenient such families, which will be especially sell suited for the use of the correlation inequalities introduced in the next section. Define, for all $A\subset\mathbb{Z}^d$,

$$\sigma_A=\prod_{j\in A}\sigma_j, n_A=\prod_{j\in A}n_j,$$

where $n_j=\frac{1}{2}(1+\sigma_j)$ is the occupation variable at j.

Lemma 5.0

Let f be local. There exist real coefficients $\hat{f}_A$ and $\tilde{f}_A$ such that both of the following representations hold:

$$f=\sum_{A\subset supp(f)}\hat{f}_A\sigma_A,\quad f=\sum_{A\subset supp(f)}\tilde{f}_A n_A.$$

Proof

We first prove the orthogonality relation of σ_A.

$$2^{-|B|}\sum_{A\subset B}\sigma_A(\tilde{\omega})\sigma_A(\omega)=I(\omega_i=\tilde{\omega}_i,\forall i\in B).\tag{7}$$

Let us first assume that $\omega_i=\tilde{\omega}_i$, for all $i\in B$. In that case, $\sigma_A(\tilde{\omega})\sigma_A(\omega)=1$, which implies (7); When there exists $i\in B$ such that $\omega_i\neq\tilde{\omega}_i$, then

$$\begin{aligned}\sum_{A\subset B}\sigma_A(\tilde{\omega})\sigma_A(\omega)&=\sum_{A\subset B\setminus\{i\}}(\sigma_A(\tilde{\omega})\sigma_A(\omega)+\sigma_{A\cup\{i\}}(\tilde{\omega})\sigma_{A\cup\{i\}}(\omega))\\&=\sum_{A\subset B\setminus\{i\}}(\sigma_A(\tilde{\omega})\sigma_A(\omega)-\sigma_A(\tilde{\omega})\sigma_A(\omega))\\&=0.\end{aligned}$$

With the orthogonality lemma (7) above,

$$\begin{aligned}
f(\omega) &= \sum_{\omega' \in \Omega_{supp(f)}} f(\omega') I(\omega_i = \omega'_i, \forall i \in supp(f)) \\
&= \sum_{\omega' \in \Omega_{supp(f)}} f(\omega') 2^{-|supp(f)|} \sum_{A \subset supp(f)} \sigma_A(\omega')\sigma_A(\omega) \\
&= \sum_{A \subset supp(f)} \{2^{-|supp(f)|} \sum_{\omega' \in \Omega_{supp(f)}} f(\omega')\sigma_A(\omega')\}\sigma_A(\omega).
\end{aligned}$$

This shows that the first identity holds with $\hat{f}_A = 2^{-|supp(f)|} \sum_{\omega' \in \Omega_{supp(f)}} f(\omega')\sigma_A(\omega')$. Since $\sigma_A = \prod_{i \in A}(2n_i - 1)$, the second identity follows from the first one.