7.13 Notes

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1 Correlation Inequalities

1.1 The GKS Inequalities

Motivation. For any point i, $h \geq 0$ implies

$$\langle \sigma_i \rangle^+_{\Lambda;\beta,h} \geq 0.$$

Similarly, knowing that the spin at some vertex j takes the value $+1$ should not decrease the probablity of observing a $+$ spin at another vertex i. That is, we expect

$$\mu^+_{\Lambda;\beta,h}(\sigma_i = 1 \mid \sigma_j = 1) \geq \mu^+_{\Lambda;\beta,h}(\sigma_i = 1)$$

which is equivalent to the positive correlation inequality:

$$\mu^+_{\Lambda;\beta,h}(\sigma_i = 1, \sigma_j = 1) \geq \mu^+_{\Lambda;\beta,h}(\sigma_i = 1)\mu^+_{\Lambda;\beta,h}(\sigma_j = 1),$$

or, in terms of expectations(we use $I_{\{\sigma_i=1\}} = \frac{1}{2}(\sigma_i + 1)$),

$$\langle \sigma_i \sigma_j \rangle^+_{\Lambda;\beta,h} \geq \langle \sigma_i \rangle^+_{\Lambda;\beta,h} \langle \sigma_j \rangle^+_{\Lambda;\beta,h}.$$

In fact, the GKS inequalities can be extended to apply under $+$, free, or periodic boundary conditions, as well as in the presence of a nonnegative external field.

Let us denote by $\boldsymbol{J} = (J_{ij})_{\{i,j\} \in \mathscr{E}^b_\Lambda}$ the collection of coupling constants, and by $\boldsymbol{h} = (h_i)$ the external field vector. As a shorthand, we write $\boldsymbol{J} \geq 0$ to indicate that $J_{ij} \geq 0$ for all $\{i,j\} \in \mathscr{E}^b_\Lambda$, and similarly $\boldsymbol{h} \geq 0$ to mean that $h_i \geq 0$ for all $i \in \Lambda$.

Given a configuration $\omega \in \Omega^\eta_\Lambda$, we define the Hamiltonian by

$$\mathscr{H}_{\Lambda;\boldsymbol{J},\boldsymbol{h}}(\omega) \coloneqq -\sum_{\{i,j\} \in \mathscr{E}^b_\Lambda} J_{ij}\, \sigma_i(\omega)\sigma_j(\omega) - \sum_{i \in \Lambda} h_i\, \sigma_i(\omega).$$

We have

Theorem 1 (GKS inequalities)**.** *Let $\boldsymbol{J}, \boldsymbol{h}$ be as above and $\Lambda \Subset \mathbb{Z}^d$. Assume that $\boldsymbol{h} \geq 0$. Then, for all $A, B \subset \Lambda$,*

$$\langle \sigma_A \rangle^+_{\Lambda;\boldsymbol{J},\boldsymbol{h}} \geq 0, \tag{1}$$

$$\langle \sigma_A \sigma_B \rangle^+_{\Lambda;\boldsymbol{J},\boldsymbol{h}} \geq \langle \sigma_A \rangle^+_{\Lambda;\boldsymbol{J},\boldsymbol{h}} \langle \sigma_B \rangle^+_{\Lambda;\boldsymbol{J},\boldsymbol{h}}. \tag{2}$$

These inequalities remain valid for $\langle \cdot \rangle^\varnothing_{\Lambda;\boldsymbol{J},\boldsymbol{h}}$ and $\langle \cdot \rangle^{\text{per}}_{\Lambda;\boldsymbol{J},\boldsymbol{h}}$.

1.1.1 Proof of the GKS Inequalities

To treat the free, $+$, and periodic boundary conditions in a unified framework, we establish the inequalities in a more general setting. Let $\Lambda \subseteq \mathbb{Z}^d$ and let $\boldsymbol{K} = (K_C)_{C \subseteq \Lambda}$ be a family of real numbers, called **coupling constants**. Consider the following probability measure on Ω_Λ:

$$\nu_{\Lambda;\boldsymbol{K}}(\omega) \coloneqq \frac{1}{Z_{\Lambda;\boldsymbol{K}}} \exp\left\{ \sum_{C \subseteq \Lambda} K_C\, \omega_C \right\},$$

where $\omega_C \coloneqq \prod_{i\in C}\omega_i$, and $Z_{\Lambda;\boldsymbol{K}}$ is the associated partition function. The Gibbs distributions $\mu^{+}_{\Lambda;\boldsymbol{J},\boldsymbol{h}}$, $\mu^{\varnothing}_{\Lambda;\boldsymbol{J},\boldsymbol{h}}$, and $\mu^{\text{per}}_{\Lambda;\boldsymbol{J},\boldsymbol{h}}$ can all be expressed in this form with $K_C \geq 0$ for all $C \subseteq \Lambda$, provided $\boldsymbol{h} \geq 0$. For instance, $\mu^{+}_{\Lambda;\beta,h} = \nu_{\Lambda;\boldsymbol{K}}$ when

$$K_C = \begin{cases} h + \beta\cdot\#\{j\notin\Lambda : j\sim i\} & \text{if } C=\{i\}\subseteq\Lambda,\\ \beta & \text{if } C=\{i,j\}\subseteq\Lambda,\ i\sim j,\\ 0 & \text{otherwise.}\end{cases}$$

It is also straightforward to verify that $\mu^{\varnothing}_{\Lambda;\beta,h}$ and $\mu^{\text{per}}_{\Lambda;\beta,h}$ can also be written in this form for suitable choices of the coefficients $\boldsymbol{K}$, which can all be taken nonnegative when $h \geq 0$.

Free boundary: The Hamiltonian is

$$H^{\varnothing}_{\Lambda}(\omega) = -\beta\sum_{i\sim j}\omega_i\omega_j - h\sum_i \omega_i.$$

Taking

$$K_C = \begin{cases} h & \text{if } C=\{i\},\\ \beta & \text{if } C=\{i,j\},\ i\sim j,\\ 0 & \text{otherwise,}\end{cases}$$

gives $\mu^{\varnothing}_{\Lambda;\beta,h} = \nu_{\Lambda;\boldsymbol{K}}$ with $K_C \geq 0$.

Periodic boundary: Same expression, but sum over torus neighbors. Define K_C the same way with $i\sim_{\text{per}} j$, and again we have $\mu^{\text{per}}_{\Lambda;\beta,h} = \nu_{\Lambda;\boldsymbol{K}}$ with $K_C \geq 0$.

Now we only need to prove the following generalization of theorem 1.

Theorem 2. *Let $\boldsymbol{K} = (K_C)_{C\subset\Lambda}$ be such that $K_C \geq 0$ for all $C\subset\Lambda$. Then for any $A, B\subset\Lambda$,*

$$\langle\sigma_A\rangle^{+}_{\Lambda;\boldsymbol{K}} \geq 0, \tag{3}$$
$$\langle\sigma_A\sigma_B\rangle^{+}_{\Lambda;\boldsymbol{K}} \geq \langle\sigma_A\rangle^{+}_{\Lambda;\boldsymbol{K}}\langle\sigma_B\rangle^{+}_{\Lambda;\boldsymbol{K}}. \tag{4}$$

Proof. By Taylor expansion, we have

$$e^{K_C\omega_C} = \sum_{n_C\geq 0}\frac{K_C^{n_C}}{n_C!}\omega_C^{n_C},$$

so that

$$\begin{aligned} Z_{\Lambda;\boldsymbol{K}}\langle\sigma_A\rangle_{\Lambda;\boldsymbol{K}} &= \sum_{\omega}\omega_A\prod_{C\subset\Lambda}e^{K_C\omega_C}\\ &= \sum_{(n_C)_{C\subset\Lambda}}\prod_{C\subset\Lambda}\frac{K_C^{n_C}}{n_C!}\sum_{\omega}\omega_A\prod_{C\subset\Lambda}\omega_C^{n_C}.\end{aligned}$$

Now observe that

$$\omega_A\prod_{C\subset\Lambda}\omega_C^{n_C} = \prod_{i\in\Lambda}\omega_i^{m_i},$$

where $m_i = \mathbf{1}_{\{i\in A\}} + \sum_{C\ni i} n_C$. It follows that

$$\sum_{\omega}\prod_{i\in\Lambda}\omega_i^{m_i} = \prod_{i\in\Lambda}\sum_{\omega_i=\pm1}\omega_i^{m_i} \geq 0,$$

which proves (3).

To prove (4), we duplicate the system and consider the product distribution

$$\nu_{\Lambda;\boldsymbol{K}}\otimes\nu_{\Lambda;\boldsymbol{K}}(\omega,\omega') \coloneqq \nu_{\Lambda;\boldsymbol{K}}(\omega)\nu_{\Lambda;\boldsymbol{K}}(\omega').$$

Define $\sigma_i(\omega,\omega') \coloneqq \omega_i$, and $\sigma'_i(\omega,\omega') \coloneqq \omega'_i$. Then,

$$\langle\sigma_A\sigma_B\rangle_{\Lambda;\boldsymbol{K}} - \langle\sigma_A\rangle_{\Lambda;\boldsymbol{K}}\langle\sigma_B\rangle_{\Lambda;\boldsymbol{K}} = \langle\sigma_A(\sigma_B-\sigma'_B)\rangle_{\nu_{\Lambda;\boldsymbol{K}}\otimes\nu_{\Lambda;\boldsymbol{K}}}.$$

We reduce the problem to showing the nonnegativity of

$$(Z_{\Lambda;\boldsymbol{K}})^2\langle\sigma_A(\sigma_B-\sigma'_B)\rangle_{\nu_{\Lambda;\boldsymbol{K}}\otimes\nu_{\Lambda;\boldsymbol{K}}}=\sum_{\omega,\omega'}\omega_A(\omega_B-\omega'_B)\prod_{C\subseteq\Lambda}e^{K_C(\omega_C+\omega'_C)}.$$

Introducing the variables $\omega''_i:=\omega_i\omega'_i$, we get

$$\begin{aligned}\sum_{\omega,\omega'}\omega_A(\omega_B-\omega'_B)\prod_C e^{K_C(\omega_C+\omega'_C)}&=\sum_{\omega,\omega''}\omega_A\omega_B(1-\omega''_B)\prod_C e^{K_C(1+\omega''_C)\omega_C}\\&=\sum_{\omega''}(1-\omega''_B)\sum_{\omega}\omega_A\omega_B\prod_C e^{K_C(1+\omega''_C)\omega_C}\\&\geq 0.\end{aligned}$$

□

1.2 The FKG Inequality

Motivation. The total order on the spin space $\{-1,1\}$ induces a natural *partial order* on the configuration space $\Omega=\{-1,1\}^{\mathbb{Z}^d}$:

$$\omega\leq\omega' \iff \omega_i\leq\omega'_i \text{ for every } i\in\mathbb{Z}^d.$$

An event $E\subset\Omega$ is called *increasing* if $\omega\in E$ and $\omega\leq\omega'$ together imply $\omega'\in E$.

If E and F are two increasing events depending only on the spins inside a finite region $\Lambda\Subset\mathbb{Z}^d$, the ferromagnetic character of the Ising model suggests that the occurrence of F can only raise the probability of E:

$$\mu^+_{\Lambda;\beta,h}(E\mid F)\ \geq\ \mu^+_{\Lambda;\beta,h}(E)\,.$$

Whenever $\mu^+_{\Lambda;\beta,h}(F)>0$, this is equivalent to the positive–correlation inequality[1]

$$\mu^+_{\Lambda;\beta,h}(E\cap F)\ \geq\ \mu^+_{\Lambda;\beta,h}(E)\,\mu^+_{\Lambda;\beta,h}(F). \tag{3.23}$$

For a local function $f:\Omega_\Lambda\to\mathbb{R}$ we write $f(\omega)\leq f(\omega')$ whenever $\omega\leq\omega'$. Such a function is said to be nondecreasing.

Example. $\sigma_i, n_i, n_A, \sum_{i\in A}n_i-n_A$ are nondecreasing.

Theorem 3 (FKG inequality)**.** *Let $\boldsymbol{J}=(J_{ij})_{i,j\in\mathbb{Z}^d}$ be a family of* non-negative *coupling constants, and $\boldsymbol{h}=(h_i)_{i\in\mathbb{Z}^d}$ an arbitrary external field. Fix a finite region $\Lambda\Subset\mathbb{Z}^d$ and an arbitrary boundary condition $\sharp$. Then, for every pair of nondecreasing local functions $f,g\colon\Omega_\Lambda\to\mathbb{R}$,*

$$\big\langle fg\big\rangle^\sharp_{\Lambda;\boldsymbol{J},\boldsymbol{h}}\ \geq\ \big\langle f\big\rangle^\sharp_{\Lambda;\boldsymbol{J},\boldsymbol{h}}\,\big\langle g\big\rangle^\sharp_{\Lambda;\boldsymbol{J},\boldsymbol{h}}. \tag{3.24}$$

♦ *A one-dimensional analogue.* Inequality (3.24) is a lattice version of the elementary fact that if $f,g\colon\mathbb{R}\to\mathbb{R}$ are nondecreasing and μ is any probability measure on $\mathbb{R}$, then

$$\langle fg\rangle_\mu\ \geq\ \langle f\rangle_\mu\,\langle g\rangle_\mu,\quad \text{since } \langle fg\rangle_\mu-\langle f\rangle_\mu\langle g\rangle_\mu=\frac{1}{2}\int(f(x)-f(y))(g(x)-g(y))\,\mu(dx)\mu(dy)\geq 0.$$

Here $f(x)-f(y)$ and $g(x)-g(y)$ share the same sign by monotonicity.

1.2.1 Proof of the FKG Inequality

Still, we provide a proof for a more general version of FKG inequality.

For $\omega=(\omega_i)$, $\omega'=(\omega'_i)$, we define

$$\omega\wedge\omega':=(\omega_i\wedge\omega'_i),\quad \omega\vee\omega':=(\omega_i\vee\omega'_i).$$

Theorem 4. *Let $\mu=\bigotimes_{i\in\Lambda}\mu_i$ be a product measure on Ω_Λ. Let $f_1,f_2,f_3,f_4:\ \Omega_\Lambda\to\mathbb{R}$ be nonnegative functions on Ω_Λ s.t.*

$$f_1(\omega)f_2(\omega')\leq f_3(\omega\wedge\omega')f_4(\omega\vee\omega'),\qquad \forall\omega,\omega'\in\Omega_A. \tag{5}$$

Then

$$\langle f_1\rangle_\mu\langle f_2\rangle_\mu\leq\langle f_3\rangle_\mu\langle f_4\rangle_\mu. \tag{6}$$

[1]Indicator functions I_E, I_F are special cases of the local functions used in the general statement below.

Theorem 4 implies FKG. Assume, without loss of generality, that the (bounded) non-negative functions f and g depend only on the spins inside $\Lambda \Subset \mathbb{Z}^d$.

For $i \in \Lambda$ and $s \in \{\pm 1\}$ set

$$\mu_i(s) \;:=\; \exp\Big\{hs + \sum_{j\notin\Lambda,\, j\sim i} J_{ij}\,\eta_j\Big\}.$$

Let

$$p(\omega) \;:=\; \frac{\exp\big(\sum_{i,j\in\Lambda,\, i\sim j} J_{ij}\,\omega_i\omega_j\big)}{Z^{\eta}_{\Lambda;\boldsymbol{J},\boldsymbol{h}}}, \qquad \mu(\omega) \;:=\; \prod_{i\in\Lambda}\mu_i(\omega_i).$$

Then

$$\langle f\rangle^{\eta}_{\Lambda;\boldsymbol{J},\boldsymbol{h}} \;=\; \sum_{\omega\in\Omega_\Lambda} f(\omega)\,p(\omega)\,\mu(\omega) \;=\; \langle fp\rangle_\mu.$$

Choose $f_1 = pf,\; f_2 = pg,\; f_3 = pf\,g,\; f_4 = p$. To obtain the FKG inequality, it suffices to verify

$$p(\omega)\,p(\omega') \;\le\; p(\omega\vee\omega')\,p(\omega\wedge\omega'), \tag{7}$$

Because $J_{ij} \ge 0$, it is enough to check

$$\omega_i\omega_j + \omega_i'\omega_j' \;\le\; (\omega_i\vee\omega_i')(\omega_j\vee\omega_j') + (\omega_i\wedge\omega_i')(\omega_j\wedge\omega_j'),$$

which is obvious if either term on the right is 1. If $\omega_i \ne \omega_i'$ and $\omega_j \ne \omega_j'$, we may assume $\omega_i = 1,\; \omega_i' = -1$; the right-hand side then equals $\omega_i\omega_j + \omega_i'\omega_j'$, so the inequality still holds.

Remark 1. *The argument uses only the ordering $\omega \le \omega' \iff \omega_i \le \omega_i'$ and not the fact that $\omega_i \in \{\pm 1\}$; it extends to any real-valued spins.*

Now we turn to prove Theorem 4.

Proof. For some fixed $i \in \Lambda$, $\omega \in \Omega_\Lambda$ can be expressed by the pair $(\tilde\omega, \omega_i)$, where $\omega \in \Omega_{\Lambda\setminus\{i\}}$. We want to show that

$$\tilde f_1(\tilde\omega)\tilde f_2(\tilde\omega') \le \tilde f_3(\tilde\omega\wedge\tilde\omega')\tilde f_4(\tilde\omega\vee\tilde\omega'), \tag{8}$$

where $\tilde f_k(\tilde\omega,\cdot) := \langle f_k(\tilde{\omega_i},\cdot)\rangle_{\mu_i} = \sum_{v=\pm1} f_k(\tilde\omega, v)\mu_i(v)$. Using the observation $|\Lambda|$ times yields the result.

The left-hand side of (8) can be written

$$\big\langle f_1(\tilde\omega,u)\,f_2(\tilde\omega',\nu)\big\rangle_{\mu_i\otimes\mu_i} = \mathbf{1}_{\{u=\nu\}}\big\langle f_1(\tilde\omega,u)\,f_2(\tilde\omega',u)\big\rangle_{\mu_i\otimes\mu_i} + \mathbf{1}_{\{u<\nu\}}\big\langle f_1(\tilde\omega,u)\,f_2(\tilde\omega',\nu) + f_1(\tilde\omega,\nu)\,f_2(\tilde\omega',u)\big\rangle_{\mu_i\otimes\mu_i}.$$

Similarly, the right-hand side of (8) becomes

$$\big\langle f_3(\tilde\omega\wedge\tilde\omega',u)\,f_4(\tilde\omega\vee\tilde\omega',\nu)\big\rangle_{\mu_i\otimes\mu_i} = \mathbf{1}_{\{u=\nu\}}\big\langle f_3(\tilde\omega\wedge\tilde\omega',u)\,f_4(\tilde\omega\vee\tilde\omega',u)\big\rangle_{\mu_i\otimes\mu_i} + \mathbf{1}_{\{u<\nu\}}\big\langle f_3(\tilde\omega\wedge\tilde\omega',u)\,f_4(\tilde\omega\vee\tilde\omega',\nu) + f_3(\tilde\omega\wedge\tilde\omega',\nu)\,f_4(\tilde\omega\vee\tilde\omega',u)\big\rangle_{\mu_i\otimes\mu_i}.$$

Hence

$$\tilde f_3(\tilde\omega\wedge\tilde\omega')\,\tilde f_4(\tilde\omega\vee\tilde\omega') - \tilde f_1(\tilde\omega)\,\tilde f_2(\tilde\omega') = \mathbf{1}_{\{u=\nu\}}\big(f_3(\tilde\omega\wedge\tilde\omega',u)\,f_4(\tilde\omega\vee\tilde\omega',\nu) - f_1(\tilde\omega,u)\,f_2(\tilde\omega',\nu)\big)\big\rangle_{\mu_n\otimes\mu_n} + \mathbf{1}_{\{u<\nu\}}\big(C+D-A-B\big)\big\rangle_{\mu_n\otimes\mu_n}, \tag{9}$$

where $A := f_1(\tilde\omega,u)f_2(\tilde\omega',\nu)$, $B := f_1(\tilde\omega,\nu)f_2(\tilde\omega',u)$, $C := f_3(\tilde\omega\wedge\tilde\omega',u)f_4(\tilde\omega\vee\tilde\omega',\nu)$, $D := f_3(\tilde\omega\wedge\tilde\omega',\nu)f_4(\tilde\omega\vee\tilde\omega',u)$.

The first term on the right of (9) is non-negative by inequality (5). To conclude we need $A+B \le C+D$.

Observe that (5) yields $A \le C$ and $B \le C$, whence

$$AB = f_1(\tilde\omega,u)\,f_2(\tilde\omega',\nu)\,f_1(\tilde\omega,\nu)\,f_2(\tilde\omega',u) \le CD.$$

If $C = 0$ this forces $A = B = 0$ and $A+B \le C+D$ is trivial. If $C \ne 0$ then

$$\frac{C+D-A-B}{C} \;=\; 1 + \frac{AB}{C^2} - \big(\tfrac{A}{C} + \tfrac{B}{C}\big) = \big(1-\tfrac{A}{C}\big)\big(1-\tfrac{B}{C}\big) \;\ge\; 0,$$

so $A+B \le C+D$ as required. $\square$

Bonus.(Fall 2024 Probability Mid-term) Let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{Ber}(p)$ and set $\boldsymbol{X} = (X_1, \dots, X_n) \in \{0,1\}^n$. Prove that for any increasing functions $f, g : \{0,1\}^n \to \mathbb{R}$,

$$\text{Cov}\big(f(\boldsymbol{X}),\, g(\boldsymbol{X})\big) \;\geq\; 0.$$

Solution. Choose $\boldsymbol{J} = 0$, $Y = 2X - 1$ and $h = \frac{1}{2}\log\frac{p}{1-p} \;\geq 0$.

1.3 Consequences of Correlation Inequalities

Now we can continue our discussion of Theorem 4.0 from Jianan Zhang's note.

Theorem 5 (from Jianan Zhang's note)**.** *Let $\beta \geq 0$ and $h \in \mathbb{R}$. Along any sequence $\Lambda_n \uparrow \mathbb{Z}^d$, the finite-volume Gibbs distributions with $+$ or $-$ boundary condition converge to infinite-volume Gibbs states $\langle \cdot \rangle^+_{\beta,h}$ and $\langle \cdot \rangle^-_{\beta,h}$. The states do not depend on the sequence (Λ_n) and are both translation invariant.*

Before we prove it, we first need some lemmas.

Lemma 1. *Let f be a nondecreasing function and $\Lambda_1 \subset \Lambda_2 \Subset \mathbb{Z}^d$. Then, for any $\beta \geq 0$ and $h \in \mathbb{R}$,*

$$\langle f \rangle^+_{\Lambda_1;\beta,h} \geq \langle f \rangle^+_{\Lambda_2;\beta,h}.$$

The same statement holds for the $-$ boundary condtion and a nonincreasing function f.

Before turning to proof, we need a spatial Markov property satisfied by $\mu^\eta_{\Lambda;\beta,h}$.

Proposition 1. *For all $\Delta \subset \Lambda \Subset \mathbb{Z}^d$, and all configurations $\eta \in \Omega$ and $\omega' \in \Omega^\eta_\Lambda$,*

$$\mu^\eta_{\Lambda;\beta,h}(\cdot \mid \sigma_i = \omega'_i, \forall I \in \Lambda \backslash \Delta) = \mu^{\omega'}_{\Delta;\beta,h}(\cdot). \tag{9}$$

Sketch. Write the finite–volume Hamiltonian with boundary η as

$$\mathscr{H}^\eta_\Lambda(\sigma) = \mathscr{H}^{\omega'}_\Delta(\sigma_\Delta) + \mathscr{H}^{\omega'}_{\Lambda\backslash\Delta}(\omega') + C(\eta, \omega'),$$

where $\mathscr{H}^{\omega'}_\Delta$ collects all interactions touching Δ (the spins on $\Lambda \setminus \Delta$ are frozen to ω'), and the other two terms do *not* depend on σ_Δ. Hence, conditioning on $\{\sigma_i = \omega'_i\}_{i\in\Lambda\backslash\Delta}$ just multiplies both numerator and denominator of the Gibbs weight by the same factor $\exp\{-\beta[\mathscr{H}^{\omega'}_{\Lambda\backslash\Delta} + C]\}$. The resulting conditional distribution is therefore the usual Gibbs measure in Δ with boundary ω', i.e. $\mu^{\omega'}_{\Delta;\beta,h}$. □

Proof of Lemma 1. It follows from (9) that

$$\begin{aligned}
\langle f \rangle^+_{\Lambda_1;\beta,h} &= \langle f \mid \sigma_i = 1,\ \forall i \in \Lambda_2 \backslash \Lambda_1 \rangle^+_{\Lambda_2;\beta,h} \\
&= \frac{\langle f I_{\{\sigma_i = 1,\ \forall i \in \Lambda_2\backslash\Lambda_1\}} \rangle^+_{\Lambda_2;\beta,h}}{\langle I_{\{\sigma_i = 1,\ \forall i \in \Lambda_2\backslash\Lambda_1\}} \rangle^+_{\Lambda_2;\beta,h}} \\
&\geq \frac{\langle f \rangle^+_{\Lambda_2;\beta,h} \langle I_{\{\sigma_i = 1,\ \forall i \in \Lambda_2\backslash\Lambda_1\}} \rangle^+_{\Lambda_2;\beta,h}}{\langle I_{\{\sigma_i = 1,\ \forall i \in \Lambda_2\backslash\Lambda_1\}} \rangle^+_{\Lambda_2;\beta,h}} = \langle f \rangle^+_{\Lambda_2;\beta,h}.
\end{aligned}$$

□

The next lemma shows that the Gibbs distributions with + and - boundary condition play an extremal role.

Lemma 2. *Let f be an arbitrary nondecreasing function. Then, for any $\beta \geq 0$ and $h \in \mathbb{R}$,*

$$\langle f \rangle^-_{\Lambda;\beta,h} \leq \langle f \rangle^\eta_{\Lambda;\beta,h} \leq \langle f \rangle^+_{\Lambda;\beta,h}.$$

Similarly, if f is a local function with $\text{supp}(f) \subset \Lambda$*, resp.* $\text{supp}(f) \subset V_N$*, then*

$$\langle f \rangle^-_{\Lambda;\beta,h} \leq \langle f \rangle^\varnothing_{\Lambda;\beta,h} \leq \langle f \rangle^+_{\Lambda;\beta,h},$$

$$\langle f \rangle^-_{V_{N-1};\beta,h} \leq \langle f \rangle^{\text{per}}_{V_N;\beta,h} \leq \langle f \rangle^+_{V_N;\beta,h}.$$

Proof. Let $I(\omega) = \exp\{\beta \sum_{\substack{i\in\Lambda, j\notin\Lambda \\ i\sim j}} \omega_i(1-\eta_j)\}$. First, observe that

$$\sum_{\omega\in\Omega_\Lambda^+} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} = \sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} I(\omega),$$

and for nondecreasing function f,

$$\sum_{\omega\in\Omega_\Lambda^+} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} f(\omega) \geq \sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} I(\omega) f(\omega).$$

This implies that

$$\langle f\rangle^+_{\Lambda;\beta,h} = \frac{\sum_{\omega\in\Omega_\Lambda^+} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} f(\omega)}{\sum_{\omega\in\Omega_\Lambda^+} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)}} \geq \frac{\sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} f(\omega) I(\omega)}{\sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} I(\omega)} = \frac{\langle If\rangle^\eta_{\Lambda;\beta,h}}{\langle I\rangle^\eta_{\Lambda;\beta,h}} \geq \langle f\rangle^\eta_{\Lambda;\beta,h}.$$

□

The proof for the free boundary condition is identical, using $I(\omega) = \exp\{\beta \sum_{\substack{i\in\Lambda, j\notin\Lambda \\ i\sim j}} \omega_i\}$.

For the periodic condtion, we have

$$\mu^{\text{per}}_{V_N;\beta,h}(\omega\mid_{V_N} \mid \sigma_i = 1 \forall i\in\Sigma_N) = \mu^+_{V_{N-1};\beta,h}(\omega),$$

where $\Sigma_N \coloneqq \{i = (i_1,\cdots,i_d)\in V_N : \exists 1\leq k\leq d \ s.t.\ i_k = 0\}$ and $\omega\mid_S \coloneqq (\omega_i)_{i\in S}$.

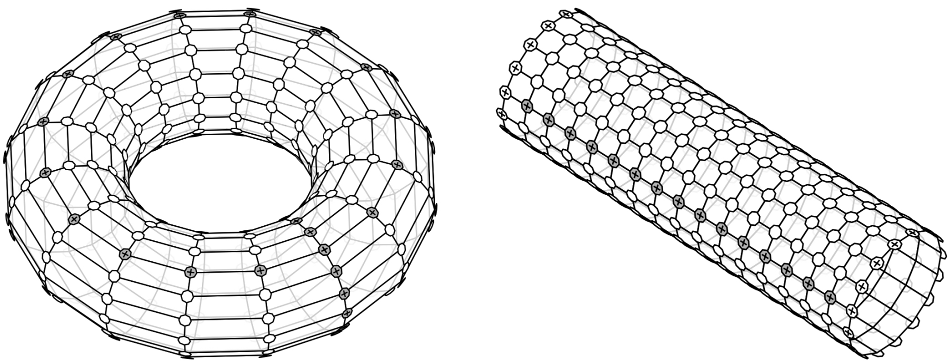

Figure 3.7: Left: The two-dimensional torus $\mathbb{T}_{16}$ with all spins along Σ_{16} forced to take the value +1. Right: opening the torus along the first "circle" of +1 yields an equivalent Ising model on a cylinder with + boundary condition and all spins forced to take the value +1 along a line. Further opening the cylinder along the line of frozen + spins yields an equivalent Ising model in the square $\{1,\ldots,15\}^2$ with + boundary condition.

Proof of Theorem 5. Let f be a local function. We have proved

$$\langle f\rangle^+_{\Lambda_n;\beta,h} = \sum_{A\subset\text{supp}(f)} \tilde{f}_A \langle n_A\rangle^+_{\Lambda_n;\beta,h}.$$

Since the functions n_A are nondecreasing, Lemma 1 implies that

$$\langle n_A\rangle^+_{\Lambda_n;\beta,h} \geq \langle n_A\rangle^+_{\Lambda_{n+1};\beta,h}.$$

By nonnegative, $\langle n_A\rangle^+_{\Lambda_n;\beta,h}$ converges as $n\to\infty$. It follows that $\langle f\rangle^+_{\Lambda_n;\beta,h}$ also has a limit $\langle f\rangle^+_{\beta,h}$.

Analogously to the subsequence selection argument in real analysis, we can easily conclude that the limit does not depend on the choice of Λ_n, and the translation invariance is also trivial.

With similar arguments, one can also construct Gibbs states using the free boundary condition. □

2 Phase Diagram

In previous sections, we have seen that infinite-volume Gibbs states can be constructed via various boundary conditions, such as + or − boundaries. The natural next question is whether these Gibbs states are identical, or if the influence of the boundary condition persists in the thermodynamic limit. This is fundamentally a question of *uniqueness* of Gibbs states.

If the limiting Gibbs state does depend on the boundary condition, then multiple distinct Gibbs states exist for the same parameters (β, h), indicating a **phase transition**. Rather than being a flaw, this lack of uniqueness is a central feature of statistical mechanics. It implies that even with full microscopic knowledge (the Hamiltonian and spin configurations), one cannot completely determine the macroscopic behavior of the system solely from the parameters β and h.

The main goal of this section is to characterize, for each pair (β, h), whether the Gibbs state is unique or not—thus establishing the *phase diagram* of the Ising model.

Definition 1. *If at least two distinct Gibbs states can be constructed for a pair (β, h), we say that there is a **first-order phase transition at** (β, h).*

We gather the corresponding claims in the form of a theorem.

Theorem 6. *1. In any $d \geq 1$, when $h \neq 0$, there is a unique Gibbs state for all values of $\beta \in \mathbb{R}_{\geq 0}$.*

2. In $d = 1$, there is a unique Gibbs state at each $(\beta, h) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$.

3. When $h = 0$ and $d \geq 2$, there exists $\beta_c = \beta_c(d) \in (0, \infty)$ such that:

- *when $\beta < \beta_c$, the Gibbs state at $(\beta, 0)$ is unique,*
- *when $\beta > \beta_c$, a first-order phase transition occurs at $(\beta, 0)$:*

$$\langle \cdot \rangle^+_{\beta,0} \neq \langle \cdot \rangle^-_{\beta,0}.$$

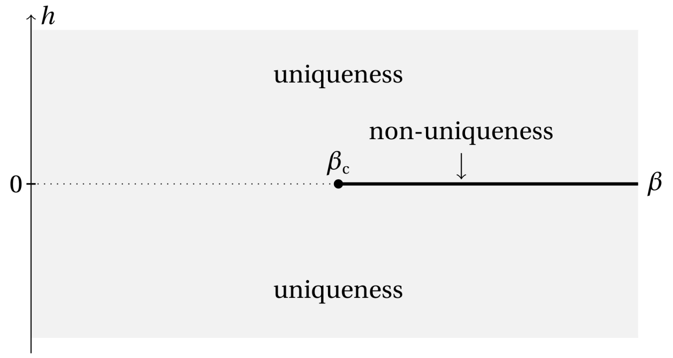

Figure 1: The phase diagram of the Ising model in $d \geq 2$.

2.1 Two Criteria for (Non)-uniqueness

In this subsection, we establish a link between uniqueness of the Gibbs state, the average magnetization density and differentiability of the pressure. We use these quantities to formulate several equivalent characterizations of uniqueness of the Gibbs state, which play a crucial role in our determination of the phase diagram.

2.1.1 A First Characterization of Uniqueness

The major role played by the states $\langle \cdot \rangle^+_{\beta,h}$ and $\langle \cdot \rangle^-_{\beta,h}$ is made clear by the following result.

Theorem 7. *Let $(\beta, h) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. The following statements are equivalent:*

1. There is a unique state at (β, h).

2. $\langle \cdot \rangle^+_{\beta,h} = \langle \cdot \rangle^-_{\beta,h}$.

3. $\langle \sigma_0 \rangle^+_{\beta,h} = \langle \sigma_0 \rangle^-_{\beta,h}$.

Proof. $1 \Rightarrow 2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 2$: By lineiarity, it suffices to prove $\langle n_A \rangle^+_{\beta,h} = \langle n_A \rangle^-_{\beta,h}$, $\forall A \Subset \mathbb{Z}^d$. Since $\sum_{i\in A} n_i - n_A$ is nondecreasing, for all k,

$$\left\langle \sum_{i\in A} n_i - n_A \right\rangle^-_{\Lambda_k;\beta,h} \leq \left\langle \sum_{i\in A} n_i - n_A \right\rangle^+_{\Lambda_k;\beta,h}.$$

Using lineiarity and translation invariance, letting $k \to \infty$and rearranging, we have

$$0 \leq \langle n_A \rangle^+_{\beta,h} - \langle n_A \rangle^+_{\beta,h} \leq \sum_{i\in A} (\langle n_i \rangle^+_{\beta,h} - \langle n_i \rangle^-_{\beta,h}) = \frac{1}{2} \sum_{i\in A} (\langle \sigma_i \rangle^+_{\beta,h} - \langle \sigma_i \rangle^-_{\beta,h}) = 0,$$

and hence $\langle n_A \rangle^+_{\beta,h} = \langle n_A \rangle^+_{\beta,h}$.

$2 \Rightarrow 1$: Applying Lemma 2 and the squeeze theorem yields

$$\langle n_A \rangle^+_{\beta,h} = \langle n_A \rangle_{\beta,h} = \langle n_A \rangle^+_{\beta,h}.$$

□

2.1.2 Some Properties of the Magnetization Density

It is natural to wonder whether the quantities of $\langle \sigma_0 \rangle^\#_{\beta,h}$ are related to the average magnetization density $m^\#_\Lambda(\beta,h) \coloneqq \langle m_\Lambda \rangle^\#_{\beta,h}$. The following result shows that they in fact coincide in the thermodynamic limit.

Proposition 2. *For any sequence $\Lambda \Uparrow \mathbb{Z}^d$, the limits*

$$m^+(\beta,h) \coloneqq \lim_{\Lambda\Uparrow\mathbb{Z}^d} m^+_\Lambda(\beta,h), \quad m^-(\beta,h) \coloneqq \lim_{\Lambda\Uparrow\mathbb{Z}^d} m^-_\Lambda(\beta,h)$$

exist and

$$m^+(\beta,h) = \langle \sigma_0 \rangle^+_{\beta,h} \quad m^-(\beta,h) = \langle \sigma_0 \rangle^-_{\beta,h}.$$

Moreover, $h \mapsto m^+(\beta,h)$ is right-continuous, while $h \mapsto m^-(\beta,h)$ is left-continuous.

Proof. By translation invariance and monotonicity,

$$\langle \sigma_0 \rangle^+_{\beta,h} = \langle m_{\Lambda_n} \rangle^+_{\beta,h} \leq \langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h}.$$

For the other bound, fix $k \geq 1$ and let $i \in \Lambda_n$. On the one hand, if $i + \mathsf{B}(k) \subset \Lambda_n$,

$$\langle \sigma_i \rangle^+_{\Lambda_n;\beta,h} \leq \langle \sigma_i \rangle^+_{i+\mathsf{B}(k);\beta,h} = \langle \sigma_0 \rangle^+_{\mathsf{B}(k);\beta,h}.$$

On the other hand, if $i + \mathsf{B}(k) \not\subset \Lambda_n$, then the box $i + B_k$ intersects $\partial^{\text{in}}\Lambda_n$. So

$$\begin{aligned}\langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h} &= \frac{1}{|\Lambda_n|} \sum_{\substack{i\in\Lambda_n:\\ i+\mathsf{B}(k)\subset\Lambda_n}} \langle \sigma_i \rangle^+_{\Lambda_n;\beta,h} + \frac{1}{|\Lambda_n|} \sum_{\substack{i\in\Lambda_n:\\ i+\mathsf{B}(k)\not\subset\Lambda_n}} \langle \sigma_i \rangle^+_{\Lambda_n;\beta,h} \\ &\leq \langle \sigma_0 \rangle^+_{\mathsf{B}(k);\beta,h} + \frac{|\mathsf{B}(k)||\partial^{\text{in}}\Lambda_n|}{|\Lambda_n|}.\end{aligned}$$

This implies that, for all $k \in \mathbb{Z}_{>0}$,

$$\limsup_n \langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h} \leq \langle \sigma_0 \rangle^+_{\mathsf{B}(k);\beta,h} \xrightarrow{k\to\infty} \langle \sigma_0 \rangle^+_{\beta,h}.$$

Now we obtain

$$\limsup_n \langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h} \leq \langle \sigma_0 \rangle^+_{\beta,h} \leq \liminf_n \langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h},$$

It follows that $m^+(\beta,h)$ exists and

$$m^+(\beta,h) = \langle \sigma_0 \rangle^+_{\beta,h}.$$

To see the one-sided continuity, we should use the FKG inequality. Fix $\Lambda \Subset \mathbb{Z}^d$.

$$\frac{\partial}{\partial h}\langle\sigma_0\rangle^+_{\Lambda;\beta,h} = \frac{\partial}{\partial h}\frac{\sum_\omega \sigma_0(\omega)\exp[\beta\sum_{i\sim j}\sigma_i\sigma_j + h\sum_i \sigma_i]}{\sum_\omega \exp[\beta\sum_{i\sim j}\sigma_i\sigma_j + h\sum_i\sigma_i]}$$
$$= \sum_i(\langle\sigma_0\sigma_i\rangle^+_{\beta,h} - \langle\sigma_0\rangle^+_{\beta,h}\langle\sigma_i\rangle^+_{\beta,h}) \geq 0,$$

so $h \mapsto \langle\sigma_0\rangle^+_{\Lambda;\beta,h}$ is nondecreasing. Letting $\Lambda \uparrow \mathbb{Z}^d$, $h \mapsto \langle\sigma_0\rangle^+_{\beta,h}$ is nondecreasing. Now let $h_m \downarrow h$ and $\Lambda_n \uparrow \mathbb{Z}^d$. So the double sequence $(\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h})_{m,n\geq 1}$ is nonincreasing and bounded.

$$\lim_{m\to\infty}\langle\sigma_0\rangle^+_{\beta,h_m} = \lim_{m\to\infty}\lim_{n\to\infty}\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h_m}$$
$$= \lim_{n\to\infty}\lim_{m\to\infty}\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h_m} = \lim_{n\to\infty}\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h} = \langle\sigma_0\rangle^+_{\beta,h}.$$

Similarly, we can prove $h \mapsto m^-(\beta,h)$ is left-continuous. □

Remark 2. *For all $h \geq 0$, $\beta \mapsto \langle\sigma_0\rangle^+_{\beta,h}$ is nondecreasing.*

The proof of Remark 2 is left as an exercise.

2.1.3 Defining the Critical Inverse Temperature

Since $\langle\sigma_0\rangle^-_{\beta,0} = -\langle\sigma_0\rangle^+_{\beta,0}$ by symmetry, so when $h = 0$, uniqueness is equivalent to $m^*(\beta) = 0$. By Remark 2, $m^*(\beta) = \langle\sigma_0\rangle^+_{\beta,0}$ is monotone in β. And naturally, we are led to the following definition.

Definition 2. *The **critical inverse temperature** is*

$$\beta_c(d) := \inf\{\beta \geq 0 : \ m^*(\beta) > 0\} = \sup\{\beta \geq 0 : \ m^*(\beta) = 0\}.$$

Remark 3. *When $m^*(\beta) > 0$, $\langle\sigma_0\sigma_i\rangle^+_{\beta,0} \geq \langle\sigma_0\rangle^+_{\beta,0}\langle\sigma_i\rangle^+_{\beta,0} = m^*(\beta)^2 > 0$.*
In particular,

$$\inf_{i\in\mathbb{Z}^d}\langle\sigma_0\sigma_i\rangle^+_{\beta,0} > 0, \qquad \forall\beta > \beta_c.$$

Such a behavior is referred to as long-range order.

2.1.4 A Second Characterization of Uniqueness

The following theorem provides the promised link between the two notions of first order transition.

Theorem 8. *For $\beta \geq 0$ and $h \in \mathbb{R}$, we have*

$$\frac{\partial\psi}{\partial h^\pm}(\beta,h) = m^\pm(\beta,h).$$

Proof. Fix $\beta \geq 0$ and $h \in \mathbb{R}$. Choose a decreasing sequence $h_k \downarrow h$ such that the pressure $\psi(\beta,\cdot)$ is differentiable at every h_k. Then

$$\frac{\partial\psi}{\partial h^+}(\beta,h) = \lim_{k\to\infty} m(\beta,h_k) = \lim_{k\to\infty} m^+(\beta,h_k) = m^+(\beta,h).$$

The identity for the left derivative follows in the same way by considering an increasing sequence $h_k \uparrow h$. □

Now,

$$m^+(\beta,h) = m^-(\beta,h) \iff \langle\sigma_0\rangle^+_{\beta,h} = \langle\sigma_0\rangle^-_{\beta,h} \iff \text{uniqueness at } (\beta,h).$$

References

[1] Friedli, Sacha, and Yvan Velenik. *Statistical mechanics of lattice systems: a concrete mathematical introduction.* Cambridge University Press, 2018.