7.27 Notes

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This note draws upon Subsections 1, 2, and 6 of Section 3.10 from the book[1].

1 Kramers–Wannier duality

In the first part (based on Subsection 3.10.1[1]), we will present a proof showing that the critical inverse temperature of the Ising model on $\mathbb{Z}^2$ is given by

$$\beta_c(2) = \frac{1}{2}\log(1+\sqrt{2}). \tag{3.63}$$

This proof is attributed to Kramers and Wannier [2].

To begin, we introduce the partition function with + boundary condition in terms of contours (for detailed derivation, refer to Equation (3.32) in [1]):

$$Z^{+}_{B(n);\beta,0} = e^{\beta|\mathscr{E}^{b}_{B(n)}|} \sum_{\omega\in\Omega^{+}_{B(n)}} \prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}. \tag{3.64}$$

Then, we define the box dual to $B(n)$ as follows

$$B(n)^* = \{-n-\frac{1}{2}, -n+\frac{1}{2}, \dots, n-\frac{1}{2}, n+\frac{1}{2}\}^2 \subset \mathbb{Z}^2_*,$$

as illustrated in the figure below.

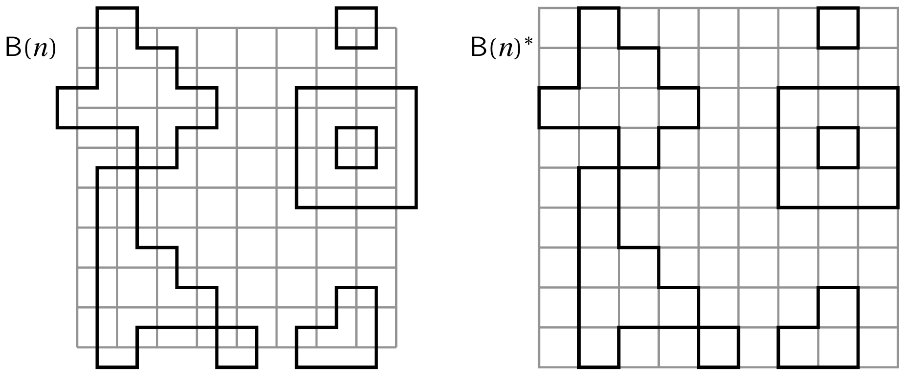

Figure 1: $B(n)$ and $B(n)^*$

Next, analogous to Equation (3.45) in [1], it can be derived that

$$\begin{aligned}
Z^{\emptyset}_{B(n)^*;\beta^*,0} &= \cosh(\beta^*)^{|\mathscr{E}_{B(n)^*}|} \sum_{E\in\mathfrak{C}^{\text{even}}_{B(n)^*}} \tanh(\beta^*)^{|E|} \sum_{\omega\in\Omega^{\Lambda}} \prod_{\{i,j\}\subset E} \omega_i\omega_j \\
&= \cosh(\beta^*)^{|\mathscr{E}_{B(n)^*}|} \sum_{E\in\mathfrak{C}^{\text{even}}_{B(n)^*}} \tanh(\beta^*)^{|E|} \prod_{i\in E} \sum_{\omega\in\Omega^{\Lambda}} \omega_i^{I(i,E)} \\
&= 2^{|B(n)^*|} \cosh(\beta^*)^{|\mathscr{E}_{B(n)^*}|} \sum_{E\in\mathfrak{C}^{\text{even}}_{B(n)^*}} \tanh(\beta^*)^{|E|}.
\end{aligned} \tag{3.65}$$

where

$$I(i,E) \overset{\text{def}}{=} \#\left\{j \in \mathbb{Z}^d : \{i,j\} \in E\right\}, \quad \mathfrak{E}^{\text{even}}_{B(n)^*} \overset{\text{def}}{=} \left\{E \in \mathscr{E}_{B(n)^*} \,\middle|\, I(i,E) \text{ is even for all } i \in \Lambda\right\}.$$

As shown in Figure 1, we will identify each set $E \in \mathfrak{E}^{\text{even}}_{B(n)^*}$ with the edges of the contours corresponding to a unique configuration $\omega \in \Omega^+_{B(n)}$

Lemma 1. *Let $E \in \mathscr{E}_{B(n)^*}$. Then $E \in \mathfrak{E}^{B(n)^*}_{even}$ if and only if E coincides with the edges of the contours of a configuration $\omega \in \Omega^{B(n)}_+$.*

Proof. If $E \in \mathscr{E}^{B(n)^*}_{\text{even}}$, we can apply the deformation operation illustrated in Figure 2.

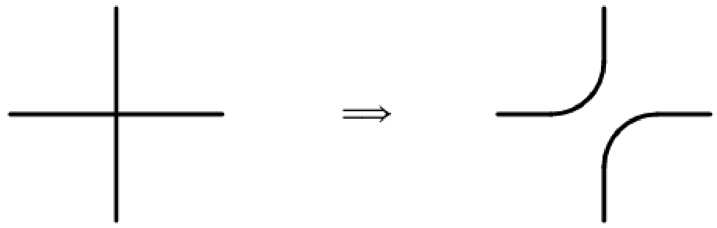

Figure 2: The deformation rule

This yields a set of disjoint closed loops, which correspond to the contours of the configuration $\omega \in \Omega^{B(n)}_+$ defined by:

$$\omega_i \overset{\text{def}}{=} (-1)^{|\{\text{loops surrounding } i\}|}, \quad i \in B(n)$$

Conversely, as noted in [1] (page 111, paragraph 2), the edge set of the contours of any configuration $\omega \in \Omega^{B(n)}_+$ belong to $\mathfrak{E}^{B(n)^*}_{\text{even}}$. □

By virtue of the previous lemma, we establish a correspondence between these sets and the contours, specifically:

$$\sum_{E \in \mathfrak{E}^{B(n)^*}_{\text{even}}} \tanh(\beta^*)^{|E|} = \sum_{\omega \in \Omega^{B(n)}_+} \prod_{\gamma \in \Gamma(\omega)} \tanh(\beta^*)^{|\gamma|}$$

Therefore, if β^* satisfies:

$$\tanh(\beta^*) = e^{-2\beta}. \tag{3.66}$$

Combine with

$$Z^+_{\Lambda;\beta,0} = e^{\beta|\mathscr{E}^b_\Lambda|} \sum_{\omega \in \Omega^\Lambda_+} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}, \tag{3.32}$$

it follows that:

$$2^{-|B(n)^*|} \cosh(\beta^*)^{-|\mathscr{E}_{B(n)^*}|} Z^{\emptyset}_{B(n)^*;\beta^*,0} = e^{-\beta|\mathscr{E}^{B(n)}_b|} Z^+_{B(n);\beta,0} \tag{3.67}$$

As $n \to \infty$:

$$\frac{|B(n)^*|}{|B(n)|} \to 1, \quad \frac{|\mathscr{E}_{B(n)^*}|}{|B(n)|} \to 2, \quad \frac{|\mathscr{E}^{B(n)}_b|}{|B(n)|} \to 2$$

Taking the logarithm of both sides of (3.67) and invoking the convergence of the pressure (for details, see Theorem 3.6 in [1]), we therefore obtain:

$$\psi(\beta,0) = \psi(\beta^*,0) - \log\sinh(2\beta^*) \tag{3.68}$$

To understand (3.68), recall that $\tanh(\beta^*) = e^{-2\beta}$. So the meaning of the (3.68) is that the pressure is essentially invariant under the transformation:

$$\beta \mapsto \beta^* = \operatorname{artanh}(e^{-2\beta}) \tag{3.69}$$

which interchanges the low and high temperatures, as can be verified in the following exercise.

Exercise 2. *Show that the mapping $\phi : x \mapsto \operatorname{artanh}(e^{-2x})$ is an involution ($\phi \circ \phi = \mathrm{id}$) with a unique fixed (self-dual) point β_{sd} equal to $\frac{1}{2}\log(1+\sqrt{2})$. Moreover, $\phi([0,\beta_{sd})) = (\beta_{sd},\infty]$.*

Finally, to prove (3.63), we present the following observations.

Since ϕ and $\log\sinh$ are both analytic on $(0,\infty)$, it follows from (3.68) that any non-analytic behavior of $\psi(\cdot,0)$ at a given inverse temperature β must necessarily imply non-analytic behavior at $\beta^* = \phi(\beta)$. Consequently, if we assume that the pressure $\psi(\cdot,0)$

1. is non-analytic at β_c,
2. is analytic everywhere else,

then β_c must satisfies $\phi\circ\phi(\beta_c)=\beta_c$, so it coincides with β_{sd}. This completes the proof of (3.63).

2 Mean-field bounds

In the second part (based on Subsection 3.10.2[1]), let $\psi_\beta^{\mathrm{CW}}(h)$, $m_\beta^{\mathrm{CW}}(h)$ and $\beta_c^{\mathrm{CW}} \stackrel{\mathrm{def}}{=} (2d)^{-1}$ represent the pressure, magnetization and critical inverse temperature of the Curie–Weiss model associated to the d-dimensional Ising model. The main theorem of this part, shows that these quantities provide rigorous bounds on the corresponding quantities for the Ising model on $\mathbb{Z}^d$.

First, let us recall some basic definitions and properties of the Curie–Weiss model.

Definition 3. *1. For a set of spins $\omega=(\omega_1,\dots,\omega_N)$ The Curie-Weiss Hamiltonian defined at inverse temperature β and with an external magnetic field h, is given by*

$$\mathscr{H}_{N;\beta,h}^{CW}(\omega) \stackrel{def}{=} -\frac{d\beta}{N}\sum_{i,j=1}^{N}\omega_i\omega_j - h\sum_{i=1}^{N}\omega_i. \tag{2.2}$$

2. We define $\Omega_N \stackrel{def}{=} \{\pm1\}^N$ as the set of all possible configurations of the Curie–Weiss model. The Gibbs distribution over Ω_N is expressed as:

$$\mu_{N;\beta,h}^{CW}(\omega) \stackrel{def}{=} \frac{e^{-\mathscr{H}_{N;\beta,h}^{CW}(\omega)}}{Z_{N;\beta,h}^{CW}}, \quad \text{where} \quad Z_{N;\beta,h}^{CW} \stackrel{def}{=} \sum_{\omega\in\Omega_N} e^{-\mathscr{H}_{N;\beta,h}^{CW}(\omega)}.$$

3. The free enregy of the Curie-Weiss model is defined by

$$f_\beta^{CW}(m) \stackrel{def}{=} -\beta d m^2 + \frac{1-m}{2}\log\frac{1-m}{2} + \frac{1+m}{2}\log\frac{1+m}{2}. \tag{2.5}$$

4. When $h\neq 0$, the supremum of $hm - f_\beta^{CW}(m)$ is attained at a unique point which we denote by $m_\beta^{CW}(h)$. This point can be written as the modified mean-field equation:

$$\tanh(2d\beta m + h) = m. \tag{2.15}$$

Theorem 4. *1. The pressure of the Curie-Weiss model*

$$\psi_\beta^{CW}(h) \stackrel{def}{=} \lim_{N\to\infty}\frac{1}{N}\log Z_{N;\beta,h}^{CW}$$

exists and is convex in h. Moreover, it equals the Legendre transform of the free energy:

$$\psi_\beta^{CW}(h) = \max_{m\in[-1,1]}\left\{hm - f_\beta^{CW}(m)\right\}. \tag{2.14}$$

2. By (2.14), the pressure can be written explicitly as

$$\psi_\beta^{CW}(h) = -d\beta\left(m_\beta^{CW}(h)\right)^2 + \log\cosh\left(2d\beta\, m_\beta^{CW}(h) + h\right) + \log 2.$$

3.

Next, we present our principal theorem of this part, which is credited to Thompson [3][4].

Theorem 5. *The following holds for the Ising model on $\mathbb{Z}^d$, $d\geq 1$:*

1. $\psi(\beta,h)\geq \psi_\beta^{CW}(h)$, for all $\beta\geq 0$ and all $h\in\mathbb{R}$;

2. $\langle\sigma_0\rangle_{\beta,h}^{+} \leq m_\beta^{CW}(h)$, for all $\beta\geq 0$ and all $h\geq 0$;

3. $\beta_c(d) \geq \beta_c^{CW}$, *for all* $d \geq 1$.

Proof. 1. Our fundamental approach is to identify a sequence of numbers that converges to $\psi(\beta, h)$ and demonstrate that each term in this sequence is no less than $\psi_\beta^{\mathrm{CW}}(h)$.

Given that the pressure functions are even with respect to h, we can assume without loss of generality that $h \geq 0$. We begin by decomposing the Hamiltonian with periodic boundary conditions:

$$\mathscr{H}^{\mathrm{per}}_{V_n;\beta,h} \stackrel{\mathrm{def}}{=} -\beta \sum_{\{i,j\} \in \mathscr{E}^{\mathrm{per}}_{\mathrm{per};V_n}} \sigma_i \sigma_j - h \sum_{i \in V_n} \sigma_i = \mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h} + \mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h},$$

where

$$\mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h} \stackrel{\mathrm{def}}{=} d\beta |V_n| m^2 - (h + 2d\beta m) \sum_{i \in V_n} \sigma_i,$$

$$\mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h} \stackrel{\mathrm{def}}{=} -\beta \sum_{\{i,j\} \in \mathscr{E}^{\mathrm{per}}_{\mathrm{per};V_n}} (\sigma_i - m)(\sigma_j - m),$$

and $m \in \mathbb{R}$ is a parameter to be determined later.

We can then express the corresponding partition function as

$$\begin{aligned} Z^{\mathrm{per}}_{V_n;\beta,h} &\stackrel{\mathrm{def}}{=} \sum_{\omega \in \Omega_{V_n}} \exp\left(-\mathscr{H}^{\mathrm{per}}_{V_n;\beta,h}(\omega)\right) \\ &= \sum_{\omega \in \Omega_{V_n}} \exp\left(-\mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h}(\omega)\right) \exp\left(-\mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h}(\omega)\right) \\ &= Z^{\mathrm{per},0}_{V_n;\beta,h} \left\langle \exp\left(-\mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h}\right) \right\rangle^{\mathrm{per},0}_{V_n;\beta,h}, \end{aligned}$$

where we introduce the Gibbs distribution

$$\mu^{\mathrm{per},0}_{V_n;\beta,h}(\omega) \stackrel{\mathrm{def}}{=} \frac{\exp\left(-\mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h}(\omega)\right)}{Z^{\mathrm{per},0}_{V_n;\beta,h}}, \quad \text{with } Z^{\mathrm{per},0}_{V_n;\beta,h} \stackrel{\mathrm{def}}{=} \sum_{\omega \in \Omega_{V_n}} \exp\left(-\mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h}(\omega)\right).$$

By the convexity of the exponential function and Jensen's inequality,

$$Z^{\mathrm{per}}_{V_n;\beta,h} \geq Z^{\mathrm{per},0}_{V_n;\beta,h} \exp\left(-\left\langle \mathscr{H}_{\mathrm{per},1;V_n;\beta,h} \right\rangle^{\mathrm{per},0}_{V_n;\beta,h}\right).$$

Notably,

$$\begin{aligned} \left\langle \mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h} \right\rangle^{\mathrm{per},0;}_{V_n;\beta,h} &= -\beta \sum_{\{i,j\} \in E_{\mathrm{per};V_n}} \left(\langle \sigma_i \rangle^{\mathrm{per},0}_{V_n;\beta,h} - m\right) \left(\langle \sigma_j \rangle^{\mathrm{per},0}_{V_n;\beta,h} - m\right) \\ &= -\beta d |V_n| \left(m - \langle \sigma_0 \rangle^{\mathrm{per},0}_{V_n;\beta,h}\right)^2. \end{aligned}$$

Since

$$\langle \sigma_0 \rangle^{\mathrm{per},0;}_{V_n;\beta,h} = \tanh(2d\beta m + h),$$

choosing m to be the largest solution to

$$m = \tanh(2d\beta m + h)$$

(i.e. $m = \psi_\beta^{\mathrm{CW}}(h)$) we get $\left\langle \mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h} \right\rangle^{\mathrm{per},0}_{V_n;\beta,h} = 0$ and, therefore,

$$Z^{\mathrm{per}}_{V_n;\beta,h} \geq Z^{\mathrm{per},0}_{V_n;\beta,h} = e^{-d\beta m^2 |V_n|} 2^{|V_n|} \cosh(2d\beta m + h)^{|V_n|}.$$

Since $\frac{1}{|V_n|} \log Z^{\mathrm{per}}_{V_n;\beta,h}$ converge to $\psi_\beta^{\mathrm{CW}}(h)$, the conclusion follows by Theorem (4).

2. Let $\Lambda = B(n)$, with $n \geq 1$, and let $i \sim 0$ denote any nearest-neighbor of the origin. Let $\langle \cdot \rangle^{+,1}_{\Lambda;\beta,h}$ Denote by $\langle \cdot \rangle^{+,1}_{\Lambda;\beta,h}$ the expectation with respect to the Gibbs distribution in Λ with no interaction between vertices 0 and i. Using the identity

$$e^{\beta\sigma_i\sigma_j} = \cosh(\beta)\left(1 + \tanh(\beta)\sigma_i\sigma_j\right), \tag{3.41}$$

we derive the following upper bound for $\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h}$:

$$\begin{aligned}\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h} &= \frac{\sum_{\omega\in\Omega^+_\Lambda} \omega_0 \exp\left\{h\sum_{j\in\Lambda}\sigma_j + \beta\sum_{\{j,k\}\in\mathscr{E}^\Lambda_b\setminus\{0,i\}}\omega_j\omega_k\right\}(1+\omega_0\omega_i\tanh\beta)}{\sum_{\omega\in\Omega^+_\Lambda} \exp\left\{h\sum_{j\in\Lambda}\sigma_j + \beta\sum_{\{j,k\}\in\mathscr{E}^\Lambda_b\setminus\{0,i\}}\omega_j\omega_k\right\}(1+\omega_0\omega_i\tanh\beta)} \\ &= \frac{\langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h} + \langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta}{1 + \langle \sigma_0\sigma_i \rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta} \leq \frac{\langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h} + \langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta}{1 + \langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h}\,\langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta},\end{aligned} \tag{3.70}$$

where we used the GKS inequality in the last inequality.

Next, observe that for any $x \geq 0$, $a \in [0,1]$, and $b \in [-1,1]$, the function

$$\frac{b + a\tanh(x)}{1 + ba\tanh(x)} \leq \frac{b + \tanh(ax)}{1 + b\tanh(ax)}. \tag{3.71}$$

holds due to the concavity of tanh and the monotonicity of $y \mapsto \frac{b+y}{1+by}$ for $y \geq 0$. Applying (3.71) to (3.70) yields

$$\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h} \leq \frac{\langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h} + \tanh\left(\beta\langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h}\right)}{1 + \langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h}\tanh\left(\beta\langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h}\right)}.$$

Using the identity

$$\frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} = \tanh(x+y),$$

this simplifies to

$$\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h} \leq \tanh\left\{\operatorname{artanh}\left(\langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h}\right) + \beta\langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h}\right\},$$

which can be rewritten as

$$\operatorname{artanh}\left(\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h}\right) \leq \operatorname{artanh}\left(\langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h}\right) + \beta\langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h}.$$

Finally, by GKS inequalities,

$$\langle \sigma_i \rangle^{+,1}_{\Lambda;\beta,h} = \langle \sigma_i e^{\beta\sigma_0\sigma_i} \rangle^{+}_{\Lambda;\beta,h} / \langle e^{\beta\sigma_0\sigma_i} \rangle^{+}_{\Lambda;\beta,h} \leq \langle \sigma_i \rangle^{+}_{\Lambda;\beta,h},$$

so that

$$\operatorname{artanh}\left(\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h}\right) \leq \operatorname{artanh}\left(\langle \sigma_0 \rangle^{+,1}_{\Lambda;\beta,h}\right) + \beta\langle \sigma_i \rangle^{+}_{\Lambda;\beta,h}. \tag{3.72}$$

Iterating (3.72) over all nearest-neighbors $i \sim 0$ successively, we derive

$$\operatorname{artanh}\left(\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h}\right) \leq \operatorname{artanh}\left(\langle \sigma_0 \rangle^{\emptyset}_{\{0\};\beta,h}\right) + \beta\sum_{i\sim 0}\langle \sigma_i \rangle^{+}_{\Lambda;\beta,h}.$$

Of course, $\langle \sigma_0 \rangle^{\emptyset}_{\{0\};\beta,h} = \tanh(h)$. Therefore,

$$\operatorname{artanh}\left(\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h}\right) \leq h + \beta\sum_{i\sim 0}\langle \sigma_i \rangle^{+}_{\Lambda;\beta,h},$$

that is,

$$\langle \sigma_0 \rangle^{+}_{\Lambda;\beta,h} \leq \tanh\left(h + \beta\sum_{i\sim 0}\langle \sigma_i \rangle^{+}_{\Lambda;\beta,h}\right).$$

Taking the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^d$ and using translation invariance $\langle \sigma_i \rangle^{+}_{\beta,h} = \langle \sigma_0 \rangle^{+}_{\beta,h}$, we obtain

$$\langle \sigma_0 \rangle^{+}_{\beta,h} \leq \tanh\left(h + 2d\beta\langle \sigma_0 \rangle^{+}_{\beta,h}\right).$$

From this we conclude $\langle \sigma_0 \rangle^{+}_{\beta,h} \leq m^{\mathrm{CW}}_\beta(h)$.

3. When $\beta < \beta^{\mathrm{CW}}_c$, the previous result implies $\langle \sigma_0 \rangle^{+}_{\beta,0} \leq m^{\mathrm{CW}}_\beta(0) = 0$. Since $\langle \sigma_0 \rangle^{+}_{\beta,0} \geq 0$, this forces $\langle \sigma_0 \rangle^{+}_{\beta,0} = 0$, proving $\beta < \beta_c(d)$.

□

As shown, Theorem (5) provides explicit bounds on the Ising model quantities on $\mathbb{Z}^d$ ($d \geq 1$) by leveraging the exact solutions of the Curie-Weiss model.

3 Random-cluster and random-current representations

In the third and final part (drawing on Subsection 3.10.6 of [1]), we will present a geometric approach to the Ising model. In the previous seminar, we covered the low-temperature and high-temperature representations during our analysis of the phase diagram. In this part, we will briefly introduce two other graphical representations of the Ising model: the random-cluster representation and the random-current representation.

3.1 Random-cluster representation

We begin with the random-cluster representation. Its starting point is analogous to the derivation of the model's high-temperature representation: we expand the Boltzmann weight in a suitable manner. Here, we express

$$e^{\beta\sigma_i\sigma_j} = e^{-\beta} + \left(e^{\beta} - e^{-\beta}\right)\mathbf{1}_{\{\sigma_i=\sigma_j\}} = e^{\beta}\left((1-p_\beta) + p_\beta \mathbf{1}_{\{\sigma_i=\sigma_j\}}\right),$$

where we define

$$p_\beta \overset{\text{def}}{=} 1 - e^{-2\beta} \in [0,1].$$

Let $\Lambda \Subset \mathbb{Z}^d$. Using the above notation, after expanding the product, we obtain

$$\prod_{\{i,j\}\in\mathscr{E}^b_\Lambda} e^{\beta\sigma_i\sigma_j} = e^{\beta|\mathscr{E}^b_\Lambda|}\sum_{E\subset\mathscr{E}^b_\Lambda} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda\setminus E|}\prod_{\{i,j\}\in E}\mathbf{1}_{\{\sigma_i=\sigma_j\}}.$$

The partition function $Z^+_{\Lambda;\beta,0}$ can thus be written as

$$\begin{aligned} Z^+_{\Lambda;\beta,0} &= e^{\beta|\mathscr{E}^b_\Lambda|}\sum_{E\subset\mathscr{E}^b_\Lambda} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda\setminus E|}\sum_{\omega\in\Omega^+_\Lambda}\prod_{\{i,j\}\in E}\mathbf{1}_{\{\sigma_i(\omega)=\sigma_j(\omega)\}} \\ &= e^{\beta|\mathscr{E}^b_\Lambda|}\sum_{E\subset\mathscr{E}^b_\Lambda} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda\setminus E|}2^{N^w_\Lambda(E)-1}, \end{aligned}$$

where $N^w_\Lambda(E)$ denotes the number of connected components (usually referred to as clusters in this context) of the graph $(\mathbb{Z}^d, E\cup\mathscr{E}_{\mathbb{Z}^d\setminus\Lambda})$. This graph is constructed by considering all vertices of $\mathbb{Z}^d$ and all edges of $\mathbb{Z}^d$ that either belong to E or do not intersect the box Λ.

The FK-percolation process in Λ with wired boundary condition is a probability distribution on $\mathcal{P}(\mathscr{E}^b_\Lambda)$ —the set of all subsets of $\mathscr{E}^b_\Lambda$. For a subset of edges $E\subset\mathscr{E}^b_\Lambda$, the probability assigned by this distribution is

$$\nu^{\text{FK},w}_{\Lambda;p_\beta,2}(E) \overset{\text{def}}{=} \frac{p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda\setminus E|}2^{N^w_\Lambda(E)}}{\sum_{E'\subset\mathscr{E}^b_\Lambda} p_\beta^{|E'|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda\setminus E'|}2^{N^w_\Lambda(E')}}.$$

For $A, B\subset\mathbb{Z}^d$, let us write $\{A\leftrightarrow B\}$ for the event that there exists a cluster intersecting both A and B.

Exercise 6. *Proceeding as above, check the following identities: for any $i,j\in\Lambda\Subset\mathbb{Z}^d$,*

$$\langle\sigma_i\rangle^+_{\Lambda;\beta,0} = \nu^{FK,w}_{\Lambda;p_\beta,2}(i\leftrightarrow\partial_{ex}\Lambda), \quad \langle\sigma_i\sigma_j\rangle^+_{\Lambda;\beta,0} = \nu^{FK,w}_{\Lambda;p_\beta,2}(i\leftrightarrow j).$$

Proof. 1. Proof of $\langle\sigma_i\rangle^+_{\Lambda;\beta,0} = \nu^{\text{FK},w}_{\Lambda;p_\beta,2}(i\leftrightarrow\partial^{\text{ex}}\Lambda)$

By definition of the expectation in the Ising model:

$$\langle\sigma_i\rangle^+_{\Lambda;\beta,0} = \frac{1}{Z^+_{\Lambda;\beta,0}}\sum_{\omega\in\Omega^+_\Lambda}\sigma_i(\omega)\exp\left(\beta\sum_{\{j,k\}\in\mathscr{E}^b_\Lambda}\sigma_j(\omega)\sigma_k(\omega)\right).$$

Using the random-cluster expansion of the Boltzmann weight:

$$\exp\left(\beta\sum_{\{j,k\}\in\mathscr{E}^b_\Lambda}\sigma_j\sigma_k\right) = e^{\beta|\mathscr{E}^b_\Lambda|}\sum_{E\subset\mathscr{E}^b_\Lambda} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda\setminus E|}\prod_{\{j,k\}\in E}\mathbf{1}_{\{\sigma_j=\sigma_k\}},$$

substituting into the expectation gives:

$$\langle \sigma_i \rangle^+_{\Lambda;\beta,0} = \frac{e^{\beta|\mathscr{E}^b_\Lambda|}}{Z^+_{\Lambda;\beta,0}} \sum_{E \subset \mathscr{E}^b_\Lambda} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda \setminus E|} \sum_{\omega \in \Omega^+_\Lambda} \sigma_i(\omega) \prod_{\{j,k\} \in E} \mathbf{1}_{\{\sigma_j(\omega)=\sigma_k(\omega)\}}.$$

For the inner sum over spins: - If i is connected to $\partial^{\text{ex}}\Lambda$ in $E \cup \mathscr{E}_{\mathbb{Z}^d \setminus \Lambda}$, all spins in the cluster are $+1$ (due to $+$ boundary conditions), so $\sigma_i = 1$. - Otherwise, the cluster containing i is isolated, and the sum over σ_i gives $\sum_{\sigma_i=\pm 1} \sigma_i = 0$.

Thus:

$$\sum_{\omega \in \Omega^+_\Lambda} \sigma_i(\omega) \prod_{\{j,k\} \in E} \mathbf{1}_{\{\sigma_j=\sigma_k\}} = 2^{N^w_\Lambda(E)-1} \mathbf{1}_{\{i \leftrightarrow \partial^{\text{ex}}\Lambda\}}.$$

Using the partition function $Z^+_{\Lambda;\beta,0} = e^{\beta|\mathscr{E}^b_\Lambda|} \sum_E p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda \setminus E|} 2^{N^w_\Lambda(E)-1}$, we simplify:

$$\langle \sigma_i \rangle^+_{\Lambda;\beta,0} = \frac{\sum_{E: i \leftrightarrow \partial^{\text{ex}}\Lambda} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda \setminus E|} 2^{N^w_\Lambda(E)}}{\sum_E p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}^b_\Lambda \setminus E|} 2^{N^w_\Lambda(E)}} = \nu^{\text{FK},w}_{\Lambda;p_\beta,2}(i \leftrightarrow \partial^{\text{ex}}\Lambda).$$

2. Proof of $\langle \sigma_i \sigma_j \rangle^+_{\Lambda;\beta,0} = \nu^{\text{FK},w}_{\Lambda;p_\beta,2}(i \leftrightarrow j)$

   The proof follows a similar line of reasoning to the previous one and thus is omitted here.

□

One feature that renders the random - cluster representation especially useful (enabling the successful import of numerous ideas and techniques developed for Bernoulli bond percolation) is the existence of an FKG inequality. Let $\Lambda \Subset \mathbb{Z}^d$ and consider the partial order on $\mathcal{P}(E_{\text{b},\Lambda})$ given by $E \leq E'$ if and only if $E \subset E'$.

From the previous exercise (6) and the Riesz–Markov–Kakutani representation theorem, one can define a probability measure $\nu^{\text{FK},w}_{p_\beta,2}$ on $\mathscr{E}$ such that

$$\nu^{\text{FK},w}_{p_\beta,2}(\mathscr{A}) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \nu^{\text{FK},w}_{\Lambda;p_\beta,2}(\mathscr{A}),$$

for all local events.

A simple yet remarkable observation is that the statements of Exercise (6) remain valid under this measure. In particular,

$$\langle \sigma_0 \rangle^+_{\beta,0} = \nu^{\text{FK},w}_{p_\beta,2}(0 \leftrightarrow \infty),$$

where $\{0 \leftrightarrow \infty\}$ corresponds to the event that there exists an infinite path of disjoint open edges starting from 0 (or, equivalently, that the cluster containing 0 has infinite cardinality).

Since Theorem 3.28 of [1] shows that the existence of a first-order phase transition at inverse temperature (and magnetic field) is equivalent to non-zero spontaneous magnetization, the above relation implies that the latter is also equivalent to percolation in the associated FK-percolation process.

3.2 Random-current representation

Next, we introduce the random-current representation. Like before, we expand the Boltzmann weight, then the product over pairs of neighbors, and finally sum over the spins. For the first step, expand the exponential as a Taylor series:

$$e^{\beta\sigma_i\sigma_j} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!}(\sigma_i\sigma_j)^n.$$

Writing $\mathbf{n} = (n_e)_{e \in \text{E}^\text{b}_\Lambda}$ for a collection of nonnegative integers, we get

$$\prod_{\{i,j\} \in \mathscr{E}^\text{b}_\Lambda} e^{\beta\sigma_i\sigma_j} = \sum_{\mathbf{n}} \left\{ \prod_{e \in \mathcal{E}^\text{b}_\Lambda} \frac{\beta^{n_e}}{n_e!} \right\} \prod_{\{i,j\} \in \mathcal{E}^\text{b}_\Lambda} (\sigma_i\sigma_j)^{n_{i,j}}.$$

The partition function $Z^+_{\Lambda;\beta,0}$ becomes

$$Z^+_{\Lambda;\beta,0} = \sum_{\mathbf{n}} \left\{ \prod_{e\in\mathscr{E}^{\rm b}_\Lambda} \frac{\beta^{n_e}}{n_e!} \right\} \sum_{\omega\in\Omega^+_\Lambda} \prod_{\{i,j\}\in\mathcal{E}^{\rm b}_\Lambda} (\sigma_i(\omega)\sigma_j(\omega))^{n_{i,j}}$$
$$= \sum_{\mathbf{n}} \left\{ \prod_{e\in\mathscr{E}^{\rm b}_\Lambda} \frac{\beta^{n_e}}{n_e!} \right\} \prod_{i\in\Lambda} \sum_{\omega_i=\pm1} \omega_i^{\hat{I}(i,\mathbf{n})},$$

where $\hat{I}(i,\mathbf{n}) \stackrel{\rm def}{=} \sum_{j:j\sim i} n_{i,j}$. Since

$$\sum_{\omega_i=\pm1} \omega_i^m = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

we conclude

$$Z^+_{\Lambda;\beta,0} = 2^{|\Lambda|} \sum_{\mathbf{n}:\partial_\Lambda\mathbf{n}=\varnothing} \prod_{e\in\mathscr{E}^{\rm b}_\Lambda} \frac{\beta^{n_e}}{n_e!} = 2^{|\Lambda|} e^{\beta|\mathscr{E}^{\rm b}_\Lambda|} \mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\varnothing),$$

where $\partial_\Lambda\mathbf{n} \stackrel{\rm def}{=} \{i\in\Lambda : \hat{I}(i,\mathbf{n}) \text{ is odd}\}$. Under the probability distribution $\mathbb{P}^+_{\Lambda;\beta}$, $\mathbf{n}=(n_e)_{e\in\mathscr{E}^{\rm b}_\Lambda}$ is a collection of independent random variables, each one distributed according to the Poisson distribution of parameter β. We will call $\mathbf{n}$ a current configuration in Λ.

Similar representations hold for arbitrary correlation functions.

Exercise 7. *Derive the following identity: for all $A \subset \Lambda \Subset \mathbb{Z}^d$,*

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=A)}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\varnothing)}.$$

The power of the random - current representation, however, lies in the fact that it also allows a probabilistic interpretation of truncated correlations in terms of various geometric events. The crucial result is the following lemma, which deals with a distribution on pairs of current configurations

$$\mathbb{P}^{+(2)}_{\Lambda;\beta}(\mathbf{n}^1,\mathbf{n}^2) \stackrel{\rm def}{=} \mathbb{P}^+_{\Lambda;\beta}(\mathbf{n}^1)\mathbb{P}^+_{\Lambda;\beta}(\mathbf{n}^2).$$

Let us denote by $i \stackrel{\mathbf{n}}{\longleftrightarrow} \partial^{\rm ex}\Lambda$ the event that there is a path connecting i to $\partial^{\rm ex}\Lambda$ along which $\mathbf{n}$ takes only positive values.

Lemma 8 (Switching Lemma)**.** *Let $\Lambda \Subset \mathbb{Z}^d$, $A\subset\Lambda$, $i\in\Lambda$ and $\mathscr{I}$ a set of current configurations in Λ. Then,*

$$\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1 = A, \partial_\Lambda\mathbf{n}^2=\{i\}, \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I})$$

$$= \mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1 = A\triangle\{i\}, \partial_\Lambda\mathbf{n}^2=\varnothing, \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}, i \stackrel{\mathbf{n}^1+\mathbf{n}^2}{\longleftrightarrow} \partial^{ex}\Lambda). \tag{3.78}$$

Proof. Define

$$w(\mathbf{n}) \stackrel{\rm def}{=} \prod_{e\in\mathscr{E}^{\rm b}_\Lambda} \frac{\beta^{n_e}}{n_e!}$$

and, for two current configurations satisfying $\mathbf{n}\le\mathbf{m}$ (that is, $n_e\le m_e$, $\forall e\in\mathscr{E}^{\rm b}_\Lambda$),

$$\binom{\mathbf{m}}{\mathbf{n}} \stackrel{\rm def}{=} \prod_{e\in\mathscr{E}^{\rm b}_\Lambda} \binom{m_e}{n_e}.$$

Change variables from the pair $(\mathbf{n}^1,\mathbf{n}^2)$ to the pair $(\mathbf{m},\mathbf{n})$ where $\mathbf{m}=\mathbf{n}^1+\mathbf{n}^2$ and $\mathbf{n}=\mathbf{n}^2$. Since $\partial_\Lambda(\mathbf{n}^1+\mathbf{n}^2) = \partial_\Lambda\mathbf{n}^1\triangle\partial_\Lambda\mathbf{n}^2$, $\mathbf{n}\le\mathbf{m}$ and

$$w(\mathbf{n}^1)w(\mathbf{n}^2) = \binom{\mathbf{n}^1+\mathbf{n}^2}{\mathbf{n}^2} w(\mathbf{n}^1+\mathbf{n}^2) = \binom{\mathbf{m}}{\mathbf{n}} w(\mathbf{m}),$$

we can rewrite

$$\sum_{\substack{\partial_\Lambda \mathbf{n}^1 = A\\ \partial_\Lambda \mathbf{n}^2=\{i\}\\ \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}}} w(\mathbf{n}^1)w(\mathbf{n}^2) = \sum_{\substack{\partial_\Lambda \mathbf{m}=A\triangle\{i\}\\ \mathbf{m}\in\mathscr{I}}} w(\mathbf{m}) \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\{i\}}} \binom{\mathbf{m}}{\mathbf{n}}. \tag{3.79}$$

Note that $i \stackrel{\mathbf{m}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda \implies i \stackrel{\mathbf{n}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda$, since $\mathbf{n}\le\mathbf{m}$. Consequently,

$$\sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\{i\}}} \binom{\mathbf{m}}{\mathbf{n}} = 0, \quad \text{when } i \stackrel{\mathbf{m}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda, \tag{3.80}$$

since $i \stackrel{\mathbf{n}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda$ whenever $\partial_\Lambda\mathbf{n}=\{i\}$. Let us therefore assume that $i \stackrel{\mathbf{m}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda$, which allows us to use the following lemma, which will be proven below. □

Lemma 9. *Let $\mathbf{m}$ be a current configuration in $\Lambda \Subset \mathbb{Z}^d$ and $C, D \subset \Lambda$. If there exists a current configuration $\mathbf{k}$ such that $\mathbf{k}\le\mathbf{m}$ and $\partial_\Lambda\mathbf{k} = C$, then*

$$\sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=D}} \binom{\mathbf{m}}{\mathbf{n}} = \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=C\triangle D}} \binom{\mathbf{m}}{\mathbf{n}}. \tag{3.81}$$

An application of this lemma with $C = D = \{i\}$ yields

$$\sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\{i\}}} \binom{\mathbf{m}}{\mathbf{n}} = \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\varnothing}} \binom{\mathbf{m}}{\mathbf{n}}, \quad \text{when } i \stackrel{\mathbf{m}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda. \tag{3.82}$$

Using (3.80) and (3.82) in (3.79), and returning to the variables $\mathbf{n}^1 = \mathbf{m}-\mathbf{n}$ and $\mathbf{n}^2=\mathbf{n}$, we get

$$\begin{aligned}\sum_{\substack{\partial_\Lambda \mathbf{n}^1 = A\\ \partial_\Lambda \mathbf{n}^2=\{i\}\\ \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}}} w(\mathbf{n}^1)w(\mathbf{n}^2) &= \sum_{\substack{\partial_\Lambda \mathbf{m}=A\triangle\{i\}\\ \mathbf{m}\in\mathscr{I}\\ i\stackrel{\mathbf{m}}{\longleftrightarrow}\partial^{\mathrm{ex}}\Lambda}} w(\mathbf{m}) \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\varnothing}} \binom{\mathbf{m}}{\mathbf{n}}\\ &= \sum_{\substack{\partial_\Lambda \mathbf{n}^1 = A\triangle\{i\}\\ \partial_\Lambda \mathbf{n}^2=\varnothing\\ \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}}} w(\mathbf{n}^1)w(\mathbf{n}^2)\,\mathbf{1}_{\{i\stackrel{\mathbf{n}^1+\mathbf{n}^2}{\longleftrightarrow}\partial^{\mathrm{ex}}\Lambda\}},\end{aligned}$$

and the proof is complete. □

Proof of Lemma (9). Let us associate to the configuration $\mathbf{m}$ the graph $G_\mathbf{m}$ with vertices $\Lambda\cup\partial^{\mathrm{ex}}\Lambda$ and with m_e edges between the endpoints of each edge $e\in\mathscr{E}^{\mathrm{b}}_\Lambda$. By assumption, $G_\mathbf{m}$ possesses a subgraph $G_\mathbf{k}$ with $\partial_\Lambda G_\mathbf{k} = C$, where $\partial_\Lambda G_\mathbf{k}$ is the set of vertices of Λ belonging to an odd number of edges.

The left - hand side of (3.81) is equal to the number of subgraphs G of $G_\mathbf{m}$ satisfying $\partial_\Lambda G = D$, while the right - hand side counts the number of subgraphs G of $G_\mathbf{m}$ satisfying $\partial_\Lambda G = C\triangle D$. But the application $G\mapsto G\triangle G_\mathbf{k}$ defines a bijection between these two families of graphs, since $\partial_\Lambda(G\triangle G_\mathbf{k}) = \partial_\Lambda G\triangle\partial_\Lambda G_\mathbf{k}$ and $(G\triangle G_\mathbf{k})\triangle G_\mathbf{k} = G$. □

As one simple application of the Switching Lemma, let us derive a probabilistic representation for the truncated 2 - point function.

Lemma 10. *For all distinct $i,j\in\Lambda\Subset\mathbb{Z}^d$,*

$$\langle\sigma_i\sigma_j\rangle^+_{\Lambda;\beta,0} = \frac{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1=\{i,j\},\partial_\Lambda\mathbf{n}^2=\varnothing, i\stackrel{\mathbf{n}^1+\mathbf{n}^2}{\longleftrightarrow}\partial^{ex}\Lambda)}{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1=\varnothing,\partial_\Lambda\mathbf{n}^2=\varnothing)}. \tag{3.83}$$

Proof. Using the representation of Exercise 7,

$$\begin{aligned}\langle\sigma_i\sigma_j\rangle^+_{\Lambda;\beta,0} &= \frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\{i,j\})}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\varnothing)} - \frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\{i\})}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\varnothing)}\cdot\frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\{j\})}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}=\varnothing)}\\ &= \frac{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1=\{i,j\},\partial_\Lambda\mathbf{n}^2=\varnothing) - \mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1=\{i\},\partial_\Lambda\mathbf{n}^2=\{j\})}{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1=\varnothing,\partial_\Lambda\mathbf{n}^2=\varnothing)}.\end{aligned}$$

Since the Switching Lemma(8) implies that

$$\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1=\{i\},\partial_\Lambda\mathbf{n}^2=\{j\}) = \mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1=\{i,j\},\partial_\Lambda\mathbf{n}^2=\varnothing, i\stackrel{\mathbf{n}^1+\mathbf{n}^2}{\longleftrightarrow}\partial^{\mathrm{ex}}\Lambda),$$

we can cancel terms in the numerator and the conclusion follows. □

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