3.8 Notes on Brownian Motion Seminar

Jieyang Hu, USTC

March 8, 2025

1 The Martingale Property of Brownian Motion

1.1 Two Useful Propositions about Continuous Martingales

Since Brownian motion is a continuous martingale, we can use the general theory of continuous martingales to study its properties.

Recall the definition of a martingale. A real-valued stochastic process $\{X(t) : t \geq 0\}$ is a **martingale** with respect to a filtration $\{\mathcal{F}(t) : t \geq 0\}$ if it is adapted to the filtration, $\mathbb{E}[|X(t)|] < \infty$ for all $t \geq 0$, and for any $0 \leq s \leq t$ we have

$$\mathbb{E}[X(t) \mid \mathcal{F}(s)] = X(s) \quad \text{a.s.}$$

If the conditional expectation satisfies $\mathbb{E}[X(t) \mid \mathcal{F}(s)] \geq X(s)$, the process is called a **submartingale**; if $\leq$ holds, it is a **supermartingale**.

In this section we focus on continuous-time martingales, while also making use of some results from discrete-time martingales. First, we list two key results for discrete-time martingales: the optional stopping theorem and Doob's maximal inequality.

Proposition 1 (Optional Sampling Theorem)**.** *If the martingale $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable, then for any stopping times $0 \leq S \leq T$ we have*

$$\mathbb{E}[X_T \mid \mathcal{F}(S)] = X_S \quad \textit{a.s.}$$

Proposition 2 (Doob's L^p Maximal Inequality)**.** *Suppose $\{X_n : n \in \mathbb{N}\}$ is a martingale or a nonnegative submartingale. Define*

$$M_n = \max_{1 \leq k \leq n} X_k$$

and let $p > 1$. Then

$$\mathbb{E}[M_n^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_n|^p].$$

We now extend these results to continuous time. Here, we focus on *continuous* martingales, meaning that almost all sample paths are continuous.

Proposition 3 (Optional Stopping Theorem)**.** *Suppose $\{X(t) : t \geq 0\}$ is a continuous martingale and $0 \leq S \leq T$ are stopping times. If the stopped process $\{X(t \wedge T) : t \geq 0\}$ is bounded by an integrable random variable (that is, there exists an integrable Y such that $|X(t \wedge T)| \leq Y$ for all $t \geq 0$), then*

$$\mathbb{E}[X(T) \mid \mathcal{F}(S)] = X(S) \quad \textit{a.s.}$$

Proof. The idea is to approximate the continuous process by a discrete one. For each fixed $N \in \mathbb{N}$, define a discrete-time martingale by

$$X_n = X(T \wedge n2^{-N})$$

and introduce stopping times

$$S' = \lfloor 2^N S \rfloor + 1 \quad \text{and} \quad T' = \lfloor 2^N T \rfloor + 1,$$

with respect to the filtration

$$\mathcal{G}_n := \mathcal{F}(n2^{-N}), \quad n \in \mathbb{N}.$$

By the discrete-time optional sampling theorem, we have

$$\mathbb{E}[X(T) \mid \mathcal{F}(S_N)] = \mathbb{E}[X_{T'} \mid \mathcal{G}(S')] = X_{S'} = X(T \wedge S_N),$$

where
$$S_N = 2^{-N}(\lfloor 2^N S \rfloor + 1).$$
Since X_n is dominated by an integrable variable, we can use the **Dominated Convergence Theorem**. For any $A \in \mathcal{F}(S)$,
$$\int_A X(T)\, d\mathbb{P} = \int_A \mathbb{E}[X(T) \mid \mathcal{F}(S_N)]\, d\mathbb{P} = \int_A \lim_{N\to\infty} X(T \wedge S_N)\, d\mathbb{P} = \int_A X(S)\, d\mathbb{P}.$$
By the definition of conditional expectation, the claim follows. □

Proposition 4 (Doob's Maximal Inequality, Continuous Version)**.** *Suppose $\{X(t) : t \geq 0\}$ is a continuous martingale and let $p > 1$. Then, for any $t \geq 0$,*
$$\mathbb{E}\left[\left(\sup_{0\leq s\leq t} |X(s)|\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X(t)|^p].$$

Proof. Fix $N \in \mathbb{N}$ and define the discrete-time martingale
$$X_n = X(tn2^{-N})$$
with respect to the filtration
$$\mathcal{G}_n := \mathcal{F}(tn2^{-N}), \quad n \in \mathbb{N}.$$
Applying the discrete version of Doob's maximal inequality gives
$$\mathbb{E}\left[\left(\sup_{1\leq k\leq 2^N} |X_k|\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X(t)|^p].$$
Taking the limit as $N \to \infty$ and using the **Monotone Convergence Theorem** yields the result. □

1.2 Brownian Motion as a Continuous Martingale

We now apply the martingale property and the optional stopping theorem to derive some results for Brownian motion. First, we introduce **Wald's lemmas**, which can be easily proved using the optional stopping theorem.

Theorem 1 (Wald's Lemmas for Brownian Motion)**.** *Let $\{B(t) : t \geq 0\}$ be a standard linear Brownian motion, and let T be a stopping time such that either*

(i) $\mathbb{E}[T] < \infty$, or

(ii) $\{B(t \wedge T) : t \geq 0\}$ is dominated by an integrable random variable.

Then, we have $\mathbb{E}[B(T)] = 0$.

Theorem 2 (Wald's Second Lemma)**.** *Let T be a stopping time for standard Brownian motion such that $\mathbb{E}[T] < \infty$. Then*
$$\mathbb{E}[B(T)^2] = \mathbb{E}[T].$$

Proof. After a simple calculation (left as an exercise), we see that $\{B(t)^2 - t : t \geq 0\}$ is a martingale. Define stopping times
$$T_n = \inf_{t\geq 0}\{|B(t)| = n\}.$$
Then,
$$|B(t \wedge T \wedge T_n)^2 - (t \wedge T \wedge T_n)| \leq n^2 + T.$$
By the optional stopping theorem, we get $\mathbb{E}[B(T \wedge T_n)^2] = \mathbb{E}[T \wedge T_n]$. Using the Dominated Convergence Theorem,
$$\mathbb{E}[B(T)^2] \geq \lim_{n\to\infty} \mathbb{E}[B(T \wedge T_n)^2] = \lim_{n\to\infty} \mathbb{E}[T \wedge T_n] = \mathbb{E}[T].$$
Applying Fatou's lemma,
$$\mathbb{E}[B(T)^2] \leq \liminf_{n\to\infty} \mathbb{E}[B(T \wedge T_n)^2] = \liminf_{n\to\infty} \mathbb{E}[T \wedge T_n] \leq \mathbb{E}[T].$$
□

This proof is reminiscent of the corresponding result in discrete-time martingales, such as the simple random walk, where the result is similar. In fact, we often draw analogies between Brownian motion and simple random walks, such as in the example below, which can also be compared to a simple random walk with absorbing barriers, yielding similar results.

Theorem 3. *Let $a < 0 < b$ and, for a standard linear Brownian motion $\{B(t) : t \geq 0\}$, define $T = \min\{t \geq 0 : B(t) \in \{a, b\}\}$. Then*

- $\mathbb{P}(B(T) = a) = \frac{b}{b-a}$ *and* $\mathbb{P}(B(T) = b) = \frac{-a}{b-a}$.
- $\mathbb{E}[T] = -ab$.

Proof. The stopping time T satisfies the conditions of the optional stopping theorem, since $|B(t \wedge T)| \leq \max\{-a, b\}$. Therefore, we have

$$\mathbb{E}[B(T)] = a\mathbb{P}(B(T) = a) + b\mathbb{P}(B(T) = b) = 0.$$

Solving this linear system gives the first result. To prove the second part, we need to check that $\mathbb{E}[T] < \infty$ in order to apply Wald's second lemma. We observe that

$$\mathbb{E}[T] \leq \sum_{k=0}^{\infty} \mathbb{P}(T > k),$$

and

$$\mathbb{P}(T > k + 1) \leq \mathbb{P}(T > k) \sup_{x \in (a,b)} \mathbb{P}_x(B(1) \in (a, b)) := \lambda \mathbb{P}(T > k),$$

where $\lambda = \sup_{x \in (a,b)} \mathbb{P}_x(B(1) \in (a, b)) < 1$. Thus, we get

$$\mathbb{E}[T] \leq \sum_{k=0}^{\infty} \mathbb{P}(T > 0)\lambda^{k-1} = \frac{1}{1 - \lambda} < \infty.$$

By Wald's second lemma, we conclude

$$\mathbb{E}[T] = \mathbb{E}[B(T)^2] = a^2\mathbb{P}(B(T) = a) + b^2\mathbb{P}(B(T) = b) = -ab.$$

□

In fact, Theorem 1 can be strengthened as follows. We omit the proof, as this result will not be used in the remainder of the seminar.

Theorem 4. *Let $\{B(t) : t \geq 0\}$ be a standard linear Brownian motion and T a stopping time such that $\mathbb{E}[T^{1/2}] < \infty$. Then $\mathbb{E}[B(T)] = 0$.*

1.3 Obtaining Martingales from Functions of Brownian Motion

We have already observed that the process $\{B(t)^2 - t : t \geq 0\}$ is a martingale. If we define a function $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = x^2$, we can derive a suitable term from $f(B(t))$. Our goal is to extend this result to a general function f. Similar to the approach used earlier, we aim to draw an analogy with simple random walk$\{S_n : n \in \mathbb{N}\}$, which resemble Brownian motion, and use this to obtain similar results. A straightforward calculation gives, for $f : \mathbb{Z} \to \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[f(S_{n+1}) \mid S_1, S_2, \ldots, S_n] - f(S_n) &= \frac{1}{2}\left(f(S_n + 1) - 2f(S_n) + f(S_n - 1)\right) \\ &= \frac{1}{2}\tilde{\Delta} f(S_n),\end{aligned}$$

where $\tilde{\Delta}$ is the second difference operator. Hence,

$$f(S_n) - \frac{1}{2}\sum_{k=0}^{n-1} \tilde{\Delta} f(S_k)$$

defines a martingale. In the Brownian motion case, one would expect a similar result, with $\tilde{\Delta} f$ replaced by the Laplacian

$$\Delta f(x) = \sum_{i=1}^{d} f_{x_i x_i},$$

and the summation replaced by an integral.

Theorem 5. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable, and $\{B(t) : t \geq 0\}$ be a d-dimensional Brownian motion. Further suppose that for all $t > 0$ and $x \in \mathbb{R}^d$, we have $\mathbb{E}_x|f(B(t))| < \infty$ and $\mathbb{E}_x \int_0^t |\Delta f(B(s))| < \infty$. Then the process $\{X(t) : t \geq 0\}$ defined by*

$$X(t) = f(B(t)) - \frac{1}{2}\int_0^t \Delta f(B(s))ds$$

is a martingale.

Proof. We calculate

$$\mathbb{E}[X(t) \mid \mathcal{F}(s)] = \mathbb{E}_{B(s)}[f(B(t-s))] - \frac{1}{2}\left(\int_0^s \Delta f(B(u))\,du - \int_0^{t-s} \mathbb{E}_{B(s)}[\Delta f(B(u))]\,du\right).$$

Noting that $\frac{1}{2}\mathfrak{p}(t,x,y) = \frac{\partial}{\partial t}\mathfrak{p}(t,x,y)$ (recall that $\mathfrak{p}(t,x,y) = (2\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{2t}\right)$), we obtain

$$\begin{aligned}
\mathbb{E}_{B(s)}[\Delta f(B(u))] &= \int_{\mathbb{R}^d} \mathfrak{p}(u, B(s), x)\Delta f(x)\,dx \\
&= \int_{\mathbb{R}^d} \Delta\mathfrak{p}(u, B(s), x) f(x)\,dx \\
&= 2\int_{\mathbb{R}^d} \frac{\partial}{\partial u}\mathfrak{p}(u, B(s), x)\,dx.
\end{aligned}$$

This follows from integration by parts. Hence,

$$\begin{aligned}
\frac{1}{2}\int_0^{t-s} \mathbb{E}_{B(s)}[\Delta f(B(u))]\,du &= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\left[\int_\varepsilon^{t-s}\frac{\partial}{\partial u}\mathfrak{p}(u, B(s), x)\,du\right] f(x)\,dx \\
&= \int_{\mathbb{R}^d} \mathfrak{p}(t-s, B(s), x) f(x)\,dx - \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\mathfrak{p}(\varepsilon, B(s), x) f(x)\,dx \\
&= \mathbb{E}_{B(s)}[f(B(t-s))] - f(B(s)),
\end{aligned}$$

which confirms that

$$\mathbb{E}[X(t) \mid \mathcal{F}(s)] = f(B(s)) - \frac{1}{2}\int_0^s \Delta f(B(u))\,du = X(s),$$

completing the proof. □

Remark 1. *Let $f(x) = x^2$. Then $\Delta f = 2$, and Theorem 5 implies that the process $\{B(t)^2 - t : t \geq 0\}$ is a martingale. Further more, if function $f : \mathbb{R}^d \to \mathbb{R}$ is harmonic, then $\{f(B(t)) : t \geq 0\}$ is a martingale. In the next section, we will explore harmonic functions in more detail.*

2 Harmonic Functions and the Dirichlet Problem

In this section, we explore the relationship between harmonic functions and Brownian motion, investigating the classical Dirichlet problem. This connection allows us to address fundamental questions about the transience and recurrence of Brownian motion, which will be discussed in the next section.

We assume that the concepts of harmonic functions, subharmonic functions, the mean value principle, the maximum modulus principle, and the uniqueness theorem for harmonic functions are already familiar.

We start by formulating the basic fact on which the relationship of Brownian motion and harmonic functions rests.

Theorem 6. *Suppose U is a domain, $\{B(t) : t \geq 0\}$ a Brownian motion started inside U and $\tau = \tau(\partial U) = \min\{t \geq 0 : \ B(t) \in \partial U\}$ the first hitting time of its boundary. Let $\varphi : \ \partial U \to \mathbb{R}$ be measurable, and such that the function $u : \ U \to \mathbb{R}$ with*

$$u(x) = \mathbb{E}_x[\varphi(B(\tau))\mathbf{1}\{\tau < \infty\}], \quad \textit{for every } x \in U$$

is locally bounded. Then u is harmonic.

Proof. The proof uses only the strong Markov property of Brownian motion. For a ball $B(x, \delta) \subset U$, let $\tilde{\tau} = \inf\{t > 0 : \ B(t) \notin B(x, \delta)\}$. The strong Markov property implies that

$$u(x) = \mathbb{E}_x[\mathbb{E}_x[\varphi(B(\tau))\mathbf{1}\{\tau < \infty\} \mid \mathcal{F}^+(\tilde{\tau})]] = \mathbb{E}_x[u(B(\tilde{\tau}))] = \frac{1}{\mathcal{L}(\partial B(x, \delta))} \int_{\partial B(x,\delta)} u(y) d(S(y)).$$

Therefore, u has the mean value property and is harmonic on U as it is locally bounded. Hence u is harmonic. □

Gauss believed that for any domain $U \subset \mathbb{R}^d$ and any continuous function $\varphi : \partial U \to \mathbb{R}$ defined on its boundary, there exists a continuous function $v : \overline{U} \to \mathbb{R}$ such that

$$\begin{cases} \Delta v = 0, & x \in U, \\ v = \varphi, & x \in \partial U. \end{cases}$$

However, this proposition is false in general; if the domain is sufficiently regular, then a solution does exist.

Definition 1. *Let $U \subset \mathbb{R}^d$ be a domain. We say that U satisfies the Poincaré cone condition at $x \in \partial U$ if there exists a cone V based at x with opening angle $\alpha > 0$, and $h > 0$ such that $V \cap B(x, h) \subset U^c$.*

Now we can state the Dirichlet Problem precisely.

Theorem 7 (Dirichlet Problem)**.** *Suppose $U \subset \mathbb{R}^d$ is a bounded domain in which every boundary point satisfies the Poincaré cone condition, and let φ be a continuous function on ∂U. Define*

$$\tau(\partial U) = \inf\{t > 0 : B(t) \in \partial U\},$$

which is an almost surely finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$ defined by

$$u(x) = \mathbb{E}_x\big[\varphi\big(B(\tau(\partial U))\big)\big], \qquad \textit{for } x \in \overline{U},$$

is the unique continuous function that is harmonic in U and satisfies $u(x) = \varphi(x)$ for all $x \in \partial U$.

Proof. Uniqueness has been established previously. Moreover, since u is bounded, it is harmonic in U by Theorem 6. It remains only to show that the Poincaré cone condition implies that u is continuous on the boundary.

Given $\varepsilon > 0$, there exists a $0 < \delta < h$ such that

$$|\varphi(y) - \varphi(z)| < \varepsilon \quad \text{for all } y \in \partial U \text{ with } |y - z| < \delta.$$

For every $x \in \overline{U}$ with $|z - x| <?$ (we will choose ? later), we have

$$|u(x) - u(z)| = \Big|\mathbb{E}_x \varphi\big(B(\tau(\partial U))\big) - \varphi(z)\Big| \leq \mathbb{E}_x \Big|\varphi\big(B(\tau(\partial U))\big) - \varphi(z)\Big|.$$

Our goal is to ensure that $\Big|\varphi\big(B(\tau(\partial U))\big) - \varphi(z)\Big|$ is sufficiently small most of the time. Note that

$$\begin{aligned} \mathbb{E}_x \Big|\varphi\big(B(\tau(\partial U))\big) - \varphi(z)\Big| &= \mathbb{E}_x\Big[\big|\varphi\big(B(\tau(\partial U))\big) - \varphi(z)\big| \, \mathbb{1}_{\{\tau(\partial B(z,\delta)) < \tau(C_z(\alpha))\}}\Big] \\ &\quad + \mathbb{E}_x\Big[\big|\varphi\big(B(\tau(\partial U))\big) - \varphi(z)\big| \, \mathbb{1}_{\{\tau(\partial B(z,\delta)) \geq \tau(C_z(\alpha))\}}\Big] \\ &\leq 2\|\varphi\|_\infty \, \mathbb{P}_x\Big(\tau(\partial B(z,\delta)) < \tau(C_z(\alpha))\Big) + \varepsilon \, \mathbb{P}_x\Big(\tau(\partial B(z,\delta)) \geq \tau(C_z(\alpha))\Big) \\ &\leq 2\|\varphi\|_\infty \, \mathbb{P}_x\Big(\tau(\partial B(z,\delta)) < \tau(C_z(\alpha))\Big) + \varepsilon. \end{aligned}$$

In order to choose ? appropriately, we require the following lemma.

Lemma 1. *Denote*

$$a = \sup_{|x|\le\frac{1}{2}} \mathbb{P}_x\Big(\tau\big(\partial B(0,1)\big) < \tau\big(C_0(\alpha)\big)\Big).$$

Then $a < 1$, and for any $k \in \mathbb{N}_+$ and $h' > 0$, we have

$$\mathbb{P}_x\Big(\tau\big(\partial B(z,h')\big) < \tau\big(C_z(\alpha)\big)\Big) \le a^k, \quad \textit{for all } x, z \in \mathbb{R}^d \textit{ with } |x-z| < 2^{-k}h'.$$

Proof of the lemma. First, we show that $a < 1$.

Without loss of generality, assume that the axis of symmetry of the cone is the x-axis and that it opens in the positive x-direction. For $x \in \overline{B(0,\frac{1}{2})} \cap C_0(\alpha)$, the probability is 0. Now, consider $x \in \overline{B(0,\frac{1}{2})} \cap C_0(\alpha)^c$. Define

$$D_1 = B(x_0, 0.8), \quad \text{where } x_0 = (0.2, 0, \dots, 0),$$

and let

$$D_2 = B(x_0, 0.1 \sin\alpha).$$

Note that

$$\overline{B(0,\tfrac{1}{2})} \subset D_1 \subset B(0,1),$$

and by continuity, $\tau(\partial D_1) \le \tau(\partial B(0,1))$. Similarly, $\tau\big(\partial(C_0(\alpha))\big) \le \tau(\partial D_2)$, so that

$$\{\tau(\partial B(0,1)) < \tau(C_0(\alpha))\} \subset \{\tau(\partial D_1) < \tau(\partial D_2)\}.$$

By Theorem 9,

$$\mathbb{P}\Big\{\tau(\partial D_1) < \tau(\partial D_2)\Big\} = \frac{u(0.8) - u(|x-x_0|)}{u(0.8) - u(0.1\sin\alpha)} \le \frac{u(0.8) - u(0.2\sin\alpha)}{u(0.8) - u(0.1\sin\alpha)} < 1,$$

which yields the desired result.

Then, if $|x| \le 2^{-k}$, by the strong Markov property,

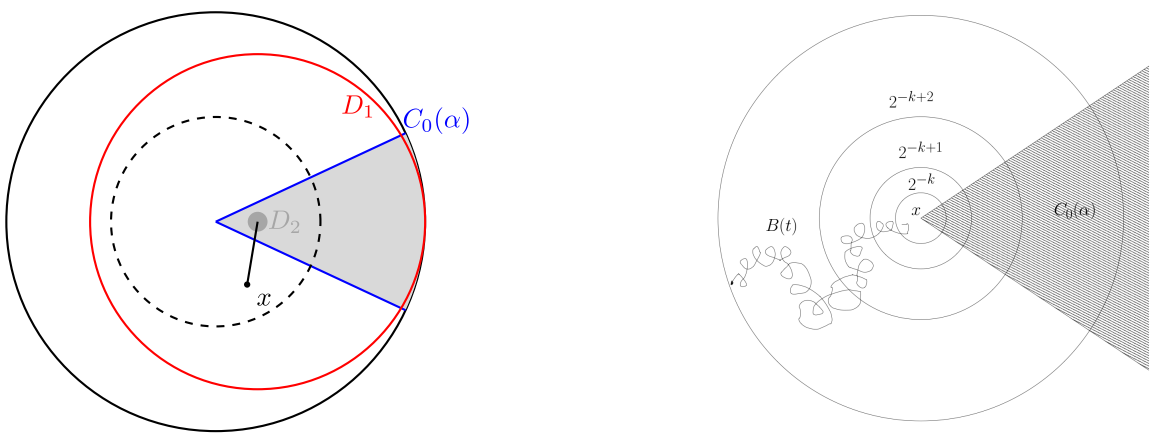
$$\mathbb{P}_x\Big(\tau(\partial B(0,1)) < \tau(C_0(\alpha))\Big) \le \prod_{i=0}^{k-1} \sup_{|x|\le 2^{i-k}} \mathbb{P}_x\Big(\tau\big(\partial B(0,2^{i+1-k})\big) < \tau\big(C_0(\alpha)\big)\Big) = a^k.$$

The conclusion follows by scaling.

Returning to the original proposition, we choose ? $= 2^{-k}\delta$ for all $k \in \mathbb{N}$. Then, by the lemma,

$$\mathbb{E}_x\Big|\varphi\big(B(\tau(\partial U))\big) - \varphi(z)\Big| \le 2\|\varphi\|_\infty a^k + \varepsilon,$$

which tends to 0 as k becomes sufficiently large. This implies that u is continuous on $\overline{U}$. □



(a) Constructing an annulus. (b) Brownian motion avoiding a cone.

Figure 1: Two figures illustrating the lemma.

Theorem 8 (Liouville's theorem)**.** *Any bounded harmonic function on $\mathbb{R}^d$ is constant.*

Proof. In the last section, we showed that if a function $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic, then

$$\mathbb{E}_x\big[u(B(t))\big] = u(x).$$

Now, consider two independent Brownian motions B_1 and B_2 starting at x and y, respectively. By symmetry, let H be the hyperplane such that reflection in H maps x to y. Define the stopping times

$$\tau_i(H) = \min\{t : B_i(t) \in H\}, \quad i = 1, 2.$$

For any fixed $t \geq 0$, we can write

$$u(x) = \mathbb{E}\Big[u(B_1(t))\, \mathbb{1}_{\{t<\tau_1(H)\}}\Big] + \mathbb{E}\Big[u(B_1(t))\, \mathbb{1}_{\{t\geq\tau_1(H)\}}\Big],$$

and

$$u(y) = \mathbb{E}\Big[u(B_2(t))\, \mathbb{1}_{\{t<\tau_2(H)\}}\Big] + \mathbb{E}\Big[u(B_2(t))\, \mathbb{1}_{\{t\geq\tau_2(H)\}}\Big].$$

Notice that the process after the stopping time has the same distribution in both cases:

$$\{B_1(t) : t \geq \tau_1(H)\} \overset{d}{=} \{B_2(t) : t \geq \tau_2(H)\}.$$

Therefore,

$$\mathbb{E}\Big[u(B_1(t))\, \mathbb{1}_{\{t\geq\tau_1(H)\}}\Big] = \mathbb{E}\Big[u(B_2(t))\, \mathbb{1}_{\{t\geq\tau_2(H)\}}\Big].$$

It follows that

$$\begin{aligned} |u(x) - u(y)| &= \Big|\mathbb{E}\Big[u(B_1(t))\, \mathbb{1}_{\{t<\tau_1(H)\}}\Big] - \mathbb{E}\Big[u(B_2(t))\, \mathbb{1}_{\{t<\tau_2(H)\}}\Big]\Big| \\ &\leq 2M\, \mathbb{P}\big(t < \tau_1(H)\big) \to 0 \quad \text{as } t \to \infty, \end{aligned}$$

where M is a bound for u. Hence, $u(x) = u(y)$ for all $x, y \in \mathbb{R}^d$, which completes the proof. □

3 Recurrence and transience of Brownian motion

A brownian motion $\{B(t) : \ t \geq 0\}$ in demension d is called transient if

$$\lim_{t\to\infty} |B(t)| = \infty \qquad \text{a.s.}$$

According to the zero-one law for tail events, such events occur with probability either zero or one. This naturally raises the question of how to determine precisely when Brownian motion is transient and when it is not.

Ultimately, the question relates to the exit probabilities of Brownian motion from an annulus. Suppose the motion starts at a point x in the annulus

$$A \coloneqq \{x \in \mathbb{R}^d : r < |x| < R\}, \quad \text{with } 0 < r < R < \infty.$$

We wish to compute the probability that the Brownian motion hits $\partial B(0, r)$ before $\partial B(0, R)$. This probability can be obtained using the fundamental solution of Laplace's equation. From classical PDE results, the function

$$u(x) = \begin{cases} |x|, & d = 1, \\ -\dfrac{1}{2\pi} \log |x|, & d = 2, \\ \dfrac{1}{d(d-2)\alpha(n)} |x|^{2-d}, & d \geq 3, \end{cases}$$

defined for $x \in \mathbb{R}^d \setminus \{0\}$, is the fundamental solution of Laplace's equation.

Note that these functions are radial, so we may write $u(r) = u(x)$ when $|x| = r$ provided there is no ambiguity. Now we define $T_r = \tau(\partial B(0, r))$ for $r > 0$, and denote $T = T_r \wedge T_R$. Hence

$$u(x) = \mathbb{E}_x[u(B(T))] = u(r)\mathbb{P}_x\{T_r < T_R\} + u(R)(1 - \mathbb{P}_x\{T_r < T_R\}).$$

This formula can be solved

$$\mathbb{P}_x(T_r < T_R) = \frac{u(R) - u(x)}{u(R) - u(r)}.$$

In conclusion, we have

Theorem 9. *Suppose $\{B(t):\ t\geq 0\}$ is a Brownian motion in dimension $d\geq 1$ started in $x\in A$, which is an open annulus A with radii $0<r<R<\infty$. Then,*

$$\mathbb{P}_x(T_r<T_R)=\begin{cases}\frac{R-|x|}{R-r}, & d=1,\\ \frac{\log R-\log|x|}{\log R-\log r}, & d=2,\\ \frac{R^{2-d}-|x|^{2-d}}{R^{2-d}-r^{2-d}}, & d\geq 3.\end{cases}$$

Let $R\to\infty$, we have

Corollary 1. *For any $x\notin B(0,r)$,*

$$\mathbb{P}_x\{T_r<\infty\}=\begin{cases}1, & d\leq 2,\\ \frac{r^{d-2}}{|x|^{d-2}}, & d\geq 3.\end{cases}$$

.

Now we turn back to the problem at the beginning of the section. We call a Markov process$\{X(t):t\geq 0\}$ with values in $\mathbb{R}^d$

- **point recurrent** if, almost surely, for every $x\in\mathbb{R}^d$ there is a (random) sequence $t_n\to\infty$ such that $X(t_n)=x$ for all $n\in\mathbb{N}$.
- **neighborhood recurrent** if, almost surely, for every $x\in\mathbb{R}^d$ and $\varepsilon>0$, there exists a (random) sequence $t_n\to\infty$ such that $X(t_n)\in B(x,\varepsilon)$ for all $n\in\mathbb{N}$.
- **transient** if it converges to infinity almost surely.

Theorem 10. *Brownian motion is*

- *point recurrent in dimension $d=1$,*
- *neighbourhood recurrent, but not point recurrent, in $d=2$,*
- *transient in dimension $d\geq 3$.*

Sketch of the proof. **Case $d=1$.** Almost surely,

$$\limsup_{t\to\infty}\frac{B(t)}{\sqrt{t}}=+\infty\quad\text{and}\quad\liminf_{t\to\infty}\frac{B(t)}{\sqrt{t}}=-\infty.$$

By the continuity of sample paths, it follows that almost surely there exists a sequence $\{t_n\}$ with $t_n\to\infty$ such that $X(t_n)=x$ for all $n\in\mathbb{N}$.

Case $d=2$. It is straightforward to see that the proposition is equivalent to showing that for every $x\in\mathbb{R}^2$ and every $r>0$, the event $\{T_r<\infty\}$ occurs almost surely. Moreover, its non-point recurrence was proved in the last seminar.

Case $d=3$. Similarly, it suffices to show that for every $0<r<|x|$,

$$\mathbb{P}_x\{T_r<\infty\}<1.$$

□

Remark 2. *Neighborhood rucurrence implies that the path of a planar Brownian motion is dense in the plane.*

3.1 The speed of escape to infinity when Brownian motion is transient

Consider a standard Brownian motion $\{B(t):\ t\geq 0\}$ in $\mathbb{R}^d$, for $d\geq 3$, and fix a sequence $t_n\uparrow\infty$. For any $\varepsilon>0$, by Fatou's lemma,

$$\mathbb{P}(|B(t_n)|<\varepsilon\sqrt{t_n}\ i.o.)\geq\limsup_{n\to\infty}\mathbb{P}(|B(t_n)|<\varepsilon\sqrt{t_n})>0.$$

By the zero-one law for tail events, the probability on the left hand side must be one. Hence

$$\liminf_{n\to\infty}\frac{|B(t)|}{\sqrt{t}}=0\qquad\text{almost surely.}\tag{1}$$

The statement is refined by the Dvoretzky-Erdös test.

Theorem 11 (Dvoretzky-Erdös test). *Let $\{B(t):\ t\geq 0\}$ be Brownian motion in $\mathbb{R}^d$ for $d\geq 3$ and $f:\ (0,\infty)\to(0,\infty)$ increasing. Then*

$$\int_1^\infty f(r)^{d-2}r^{-d/2}dr<\infty \qquad if\ and\ only\ if \qquad \liminf_{t\to\infty}\frac{|B(t)|}{f(t)}=\infty\ almost\ surely.$$

Conversely, if the integral diverges, then $\liminf_{t\to\infty}|B(t)|/f(t)=0$ *almost surely.*

To prove this, we shall first introduce two useful lemmas in discrete probability.

Lemma 2 (Paley-Zygmund inequality, second moment method). *For any nonnegative variable X with $\mathbb{E}[X^2]<\infty$,*

$$\mathbb{P}(X>0)\geq\frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

Proof. The Cauchy-Schwarz inequality gives

$$\mathbb{E}[X]=\mathbb{E}[X\mathbb{1}_{\{X>0\}}]\leq\mathbb{E}[X^2]^{1/2}(\mathbb{P}(X>0))^{\frac{1}{2}},$$

and the required inequality follows immediately. □

Lemma 3 (Kochen-Stone lemma). *Suppose $E_1,E_2,\cdots$ are events with*

$$\sum_{n=1}^\infty\mathbb{P}(E_n)=\infty\ and\ \liminf_{k\to\infty}\frac{\sum_{m,n=1}^k\mathbb{P}(E_n\cap E_m)}{(\sum_{n=1}^k\mathbb{P}(E_n))^2}<\infty.$$

Then, with positive probability, infinity many of the events take place.

Lemma 3 is a direct corollary from lemma 2.
A core estimate in the proof of the Dvoretzky-Erdős test is given by the following lemma.

Lemma 4. *There exists a constant $C_1>0$, depending only on the dimension d, such that for any $\rho>0$ we have*

$$\sup_{x\in\mathbb{R}^d}\mathbb{P}_x\Big(there\ exists\ t>1\ with\ \big|B(t)\big|\leq\rho\Big)\leq C_1\,\rho^{d-2}.$$

Proof. Using Corollary 1, we obtain

$$\begin{aligned}\mathbb{P}_x\Big(\text{there exists } t>1 \text{ with } \big|B(t)\big|\leq\rho\Big)&\leq\mathbb{E}_0\left[\left(\frac{\rho}{\big|B(1)+x\big|}\right)^{d-2}\right]\\&\leq\rho^{d-2}\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\big|y+x\big|^{2-d}\exp\Big(-\frac{|y|^2}{2}\Big)dy\\&:=C_1\,\rho^{d-2},\end{aligned}$$

where

$$C_1=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\big|y+x\big|^{2-d}\exp\Big(-\frac{|y|^2}{2}\Big)dy\in(0,\infty)$$

depends only on d. □

Now we can prove Theorem 11.
Proof of Theorem 11. First, define the events

$$A_n=\Big\{\text{there exists } t\in\big(2^n,2^{n+1}\big] \text{ with } \big|B(t)\big|\leq f(t)\Big\}.$$

By Brownian scaling, the monotonicity of f, and Lemma 4, we have

$$\begin{aligned}\mathbb{P}(A_n)&\leq\mathbb{P}\Big(\text{there exists } t>1 \text{ with } \big|B(t)\big|\leq f(2^{n+1})\,2^{-n/2}\Big)\\&\leq C_1\,\Big(f(2^{n+1})\,2^{-n/2}\Big)^{d-2}.\end{aligned}$$

Thus, the convergence of

$$\sum_{n=1}^{\infty}\Big(f(2^{n+1})\,2^{-n/2}\Big)^{d-2}<\infty \tag{2}$$

is equivalent to the statement that, almost surely, the set $\{|B(t)| \le f(t)\}$ is bounded. It follows that

$$\liminf_{t\to\infty} |B(t)| \le f(t) \quad \text{a.s.,}$$

since (2) also holds for any constant multiple of f in place of f.

Conversely, suppose that the series diverges, that is,

$$\sum_{n=1}^{\infty}\Big(f(2^{n+1})\,2^{-n/2}\Big)^{d-2}=\infty.$$

Recalling (1), we may assume that $f(t) < \sqrt{t}$ for all sufficiently large t, and hence for all $t > 0$.

Fix $\rho \in (0,1)$. Our aim is to prove that $\sum_n \mathbb{P}(A_n) = \infty$. Define

$$I_\rho = \int_1^2 \mathbb{1}_{\{|B(t)|\le\rho\}}\,dt.$$

If we choose $\rho = f(2^n)\,2^{-n/2}$, then

$$\mathbb{P}(A_n) \ge \mathbb{P}(I_\rho > 0) \ge \frac{\mathbb{E}[I_\rho]^2}{\mathbb{E}[I_\rho^2]}.$$

Hence we shall estimate $\mathbb{E}[I_\rho]$ and $\mathbb{E}[I_\rho^2]$ respectively.

First we have

$$C_2\rho^d \le \mathbb{E}[I_\rho] \le C_3\rho^d$$

for suitable C_2, C_3 depending only on d. To complement this by an estimate of the second moment,

$$\begin{aligned}\mathbb{E}[I_n^2] &= 2\mathbb{E}\left[\int_1^2 \mathbb{1}_{\{|B(t)|\le\rho\}}\int_t^2 \mathbb{1}_{\{|B(s)|\le\rho\}}\,ds\,dt\right]\\ &\le 2\mathbb{E}\left[\int_1^2 \mathbb{1}_{\{|B(t)|\le\rho\}}\mathbb{E}_{B(t)}\int_0^\infty \mathbb{1}_{\{|\tilde{B}(t)|\le\rho\}}\,ds\,dt\right],\end{aligned}$$

where Brownian motion $\{\tilde{B}(t): \ t \ge 0\}$ starts at $B(t)$.

Given $x > 0$, let $T = \inf_{t>0}\{|B(t)| = x\}$ and use the strong Markov property to see that

$$\mathbb{E}_0\int_0^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\,ds \ge \mathbb{E}_0\int_T^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\,ds = \int_x^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\,ds.$$

Hence we obtain

$$\mathbb{E}[I_\rho^2] \le 2C_3\rho^d\mathbb{E}_0\int_0^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\,ds.$$

Moreover, by Brownian scaling,

$$\mathbb{E}_0\int_0^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\,ds = \rho^2\mathbb{E}_0\int_0^\infty \mathbb{1}_{\{|B(s)|\le1\}}\,ds \le \rho^2\left(1+\int_1^\infty \frac{\mathcal{L}(B(0,1))}{(2\pi s)^{d/2}}\right)ds = C_4\rho^2,$$

and in summary, $\mathbb{E}[I_\rho^2] \le 2C_3C_4\rho^{d+2}$. Now we see that

$$\mathbb{P}(A_n) \ge C_5\left(f(2^b)2^{-n/2}\right)^{d-2},$$

so $\sum_n \mathbb{P}(A_n) < \infty$. It remains to show that

$$\liminf_{k\to\infty}\frac{\sum_{m,n=1}^{k}\mathbb{P}(A_m\cap A_n)}{\sum_{k=1}^{n}\mathbb{P}(A_n)^2} = 2\liminf_{k\to\infty}\frac{\sum_{m=1}^{k}\mathbb{P}(A_m)\sum_{n=m+2}^{k}\mathbb{P}(A_n \mid A_m)}{\sum_{k=1}^{n}\mathbb{P}(A_n)^2} < \infty.$$

By Brownian scaling, we have

$$\mathbb{P}[A_n \mid A_m] \le \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\text{there exists } t > 1 \text{ with } |B(t)| \le f(2^{n+1})2^{(1-n)/2}) \le C_1 \left(f(2^{n+1})2^{(1-n)/2}\right)^{d-2}.$$

Finally, using the assumption that $f(t) < \sqrt{t}$, we get that

$$\liminf_{k\to\infty} \frac{\sum_{m=1}^k \mathbb{P}(A_m) \sum_{n=m+2}^k \mathbb{P}(A_n \mid A_m)}{\sum_{k=1}^n \mathbb{P}(A_n)^2} \le 2\frac{C_1}{C_5} \liminf_{k\to\infty} \frac{\sum_{n=1}^k (f(2^{n+1})2^{(1-n)/2})^{d-2}}{\sum_{n=1}^k (f(2^n)2^{-n/2})^{d-2}} < \infty. \tag{3}$$

The Kochen-Stone lemma yields that $\mathbb{P}\{A_n \text{ i.o.}\} > 0$, and hence $\mathbb{P}\{A_n \text{ i.o.}\} = 1$ because it is a tail event. It follows that $\liminf_{t\to\infty} |B(t)|/f(t) = 0$ immediately since (2) also holds for any constant multiple of f in place of f.Given $x > 0$, let

$$T = \inf\{t > 0 : |B(t)| = x\}.$$

Using the strong Markov property, we have

$$\mathbb{E}_0 \int_0^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\, ds \ge \mathbb{E}_0 \int_T^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\, ds = \int_x^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\, ds.$$

Hence, we obtain

$$\mathbb{E}[I_\rho^2] \le 2C_3\rho^d\, \mathbb{E}_0 \int_0^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\, ds.$$

Moreover, by Brownian scaling,

$$\begin{aligned}\mathbb{E}_0 \int_0^\infty \mathbb{1}_{\{|B(s)|\le\rho\}}\, ds &= \rho^2\, \mathbb{E}_0 \int_0^\infty \mathbb{1}_{\{|B(s)|\le 1\}}\, ds \\ &\le \rho^2 \left(1 + \int_1^\infty \frac{\mathcal{L}(B(0,1))}{(2\pi s)^{d/2}}\, ds\right) = C_4\, \rho^2,\end{aligned}$$

so that in summary,

$$\mathbb{E}[I_\rho^2] \le 2C_3C_4\, \rho^{d+2}.$$

Now, we deduce that

$$\mathbb{P}(A_n) \ge C_5 \left(f(2^{n+1})\, 2^{-n/2}\right)^{d-2},$$

so that $\sum_n \mathbb{P}(A_n) < \infty$. It remains to show that

$$\liminf_{k\to\infty} \frac{\sum_{m,n=1}^k \mathbb{P}(A_m \cap A_n)}{\sum_{n=1}^k \mathbb{P}(A_n)^2} = 2 \liminf_{k\to\infty} \frac{\sum_{m=1}^k \mathbb{P}(A_m) \sum_{n=m+2}^k \mathbb{P}(A_n \mid A_m)}{\sum_{n=1}^k \mathbb{P}(A_n)^2} < \infty.$$

By Brownian scaling, we have

$$\mathbb{P}\Big[A_n \mid A_m\Big] \le \sup_{x\in\mathbb{R}^d} \mathbb{P}_x\Big(\text{there exists } t > 1 \text{ with } |B(t)| \le f(2^{n+1})\, 2^{(1-n)/2}\Big) \le C_1\Big(f(2^{n+1})\, 2^{(1-n)/2}\Big)^{d-2}.$$

Finally, using the assumption that $f(t) < \sqrt{t}$, we obtain

$$\liminf_{k\to\infty} \frac{\sum_{m=1}^k \mathbb{P}(A_m) \sum_{n=m+2}^k \mathbb{P}(A_n \mid A_m)}{\sum_{n=1}^k \mathbb{P}(A_n)^2} \le 2\frac{C_1}{C_5} \liminf_{k\to\infty} \frac{\sum_{n=1}^k \Big(f(2^{n+1})\, 2^{(1-n)/2}\Big)^{d-2}}{\sum_{n=1}^k \Big(f(2^n)\, 2^{-n/2}\Big)^{d-2}} < \infty. \tag{4}$$

By the Kochen-Stone lemma, it follows that

$$\mathbb{P}\{A_n \text{ i.o.}\} > 0,$$

and since $\{A_n\}$ is a tail event, we have $\mathbb{P}\{A_n \text{ i.o.}\} = 1$. Consequently,

$$\liminf_{t\to\infty} \frac{|B(t)|}{f(t)} = 0,$$

since the bound in (4) holds for any constant multiple of f in place of f.

References

[1] Peter Mörters and Yuval Peres, *Brownian Motion*, Cambridge University Press.